WAVELETS ON LOCAL FIELDS

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ABSTRACT. A local field is a locally compact topological field with respect to a non-discrete topology. Given such a field, one can define an absolute value on it. A structure theorem on local fields states that any local field is isomorphic to one of the following types.

- Characteristic zero : Either $\mathbb{R}$ or $\mathbb{C}$ or a finite extension of the p-adic number field $\mathbb{Q}_p$.
- Finite characteristic : A field of formal Laurent series over a finite field $\mathbb{F}_q$.

On $\mathbb{R}$ and $\mathbb{C}$, a function ψ in $L^2(\mathbb{R})$ is called an "orthonormal wavelet" if suitable translates and dialates of ψ give an orthonormal basis for $L^2(\mathbb{R})$. Real wavelet analysis has proven to be an area of mathematics which has considerable applications to audio signals and image processsing.

A similar definition of wavelets can now be made for the other local fields in the above list. In this project, I propose to understand local fields and their structure and also try to understand how one can define wavelets and frames on them and also understand their properties.

1. LOCAL FIELDS

~~We start off by understanding the structure and a few properties of local fields. The main results we shall be looking at will be the structure theorem for local fields, rings of integers in local fields and the Fourier analysis on local fields. Before we proceed in that direction, first let us look at some examples of local fields.~~

1.1. p-adic and p-series fields.

For an integer $n \in \mathbb{Z}$, we define its p-adic norm $|.|_p$ as follows. If $n = 0$, then $|n|_p = 0$. If $n \neq 0$, then we write $n = p^k l$, where k and l are integers and l is relatively prime to p, and define $|n|_p = p^k$. We can easily verify that $|.|_p$ is a norm on the integers which satisfies the stronger triangle inequality $|m + n|_p \leq \max(|m|_p, |n|_p)$ for $m, n \in \mathbb{Z}$. If we use the usual arithmetic for the integers and define a metric by $d(m, n) = |m - n|_p$, then $\mathbb{Z}$ is a metric space which is not complete. Its completion with respect to the above metric is called the p-adic integers. Its field of quotients is called the p-adic numbers.

The p-adic numbers can also be obtained directly by extending the definition of the p-adic norm to the rational numbers in a natural way (i.e., write $m/n = p^k r/s$ with r and s relatively prime to p and define $|m/n|_p = p^{-k}$) and then completing the rationals as a metric space with respect to the induced metric. In either case, we obtain a totally disconnected locally compact topological field of characteristic zero.

The elements of this field are identified as formal Laurent series:

$$x = \sum_{k=l}^{\infty} a_k p^k$$

, where where $a_k \in \{0, 1, 2, , p1\}$ and we 'carry' in the arithmetic from left to right. For example, let $p = 7$, $x = 3 + 4.7 + 3.7^2$ and $y = 5 + 3.7 + 5.7^2$, then $x + y = 1 + 1.7 + 0.7^2 + 1.7^3$. Multiplication is also similar.

This series is called the *p-adic field* and is denoted $\mathbb{Q}_p$.

Now, we again consider the same set of formal Laurent series, but do the addition and multiplication modulo p. For example, if $p = 7$,$x = 3 + 4.7 + 3.7^2$ and $y = 5 + 3.7 + 5.7^2$, then $x + y = 1 + 1.7^2$. If we use the same norm and metric as used for $\mathbb{Q}_p$, then we obtain another totally disconnected locally compact topological field. This field is of characteristic p and is called *p-series* field.

1.2. **The general local field.**

We will now discuss general local fields. Let K be a field and a topological space. Then K is called a locally compact field or a local field if both K^+ and $K^\star$ are locally compact abelian groups, where K^+ and $K^\star$ denote the additive and multiplicative groups of K respectively. If K is any field and is endowed with the discrete topology, then K is a local field. So we will only consider non-discrete fields.

Theorem 1.1 (Structure theorem for local fields)**.** *Let K be a locally compact, non-discrete and topologically complete field.*

1. *If K is connected then K is either $\mathbb{R}$ or $\mathbb{C}$.*
2. *If K is not connected then K is totally disconnected.*
   - *If K is of finite characteristic, then K is a field of formal power series over a finite field $\mathbb{F}_{p^c}$. If $c = 1$, then K is a p-series field. If $c \neq 1$ then K is an algebraic extension of degree c of a p-series field.*
   - *If K is of characteristic zero then K is either a p-adic number field or a finite algebraic extension of such a field.*

From now on, a local field K is always a locally compact, non-discrete, totally disconnected field. A proof of the above theorem can be found in [4]. Proofs of most assertions in the sequel are available in [1].

1.3. **Properties of local fields.**

Let K be a local field. Since K^+ is a locally compact abelian group, we choose a Haar measure dx for K^+. If $\alpha \neq 0$ $\alpha \in K$, then $d(\alpha x)$ is also a Haar measure. Let $d(\alpha x) = |\alpha| dx$.

We call $|\alpha|$ the **absolute value** or **valuation** of α. We also let $|0| = 0$.

The map $x \to |x|$ has the following properties:

1. $|x| = 0$ if and only if $x = 0$;
2. $|xy| = |x||y|$ for all $x, y \in K$;
3. $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$.

Property (3) is called the **ultrametric inequality**. It follows from this property that

$$|x+y| = \max\{|x|, |y|\} \text{ if } |x| \neq |y|$$

The set $\mathfrak{D} = \{x \in K : |x| \leq 1\}$ is called the *ring of integers* in K. It is the unique maximal compact subring of K. This is because $\mathfrak{D}$ is a ball of radius 1 and so is compact. By the (3) and (2) above, $\mathfrak{D}$ is clearly a ring. If S is a relatively compact subset of K, then $a \in S$ implies that the sequence $\{a^n\}$ has an accumulation point in S, and so $|a|_p \leq 1$. Thus $S \subseteq \mathfrak{D}$ and $\mathfrak{D}$ is indeed maximal.

Example 1.2. If K is the 2-adic number field then $\mathfrak{D}$ is the ring of 2-adic integers.

Define $\mathfrak{P} = \{x \in K : |x| < 1\}$. The set $\mathfrak{P}$ is called the *prime ideal* in K. The prime ideal in K is the unique maximal ideal in $\mathfrak{D}$. It is principal and prime. Since K is totally disconnected, the set of values $|x|$, as x varies over K, is a discrete set of the form $\{s^k : k \in \mathbb{Z}\} \cup \{0\}$ for some $s > 0$. Hence, there is an element of $\mathfrak{P}$ of maximal absolute value. Let $\mathfrak{p}$ be a fixed element of maximum absolute value in $\mathfrak{p}$. Such an element is called a *prime element* of K. Note that as an ideal in $\mathfrak{D}$, $\mathfrak{P} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$.

Example 1.3. If $K = \mathbb{Q}_2$, then $x \in \mathfrak{D}$ iff $x = \sum_{k=0}^{\infty} a_k 2^k$ with $a_k = 0$ or 1 ; also, $x \in \mathfrak{P}$ iff $x = \sum_{k=1}^{\infty} a_k 2^k$ with $a_k = 0$ or 1 and $\mathfrak{p} = 2$. All these follow from definitions and the ultrametric inequality.

It can be proved that $\mathfrak{D}$ is compact and open. Hence $\mathfrak{P}$ is compact and open. Since $\mathfrak{P}$ is compact, $\mathfrak{D}/\mathfrak{P}$ is compact. Since $\mathfrak{P}$ is open, $\mathfrak{D}/\mathfrak{P}$ is discrete. Also, since $\mathfrak{P}$ is a maximal ideal, in $\mathfrak{D}$, we have $\mathfrak{D}/\mathfrak{P}$ is a field. Thus, $\mathfrak{D}/\mathfrak{P}$ is isomorphic to a finite field $\mathbb{F}_q$, where $q = p^c$ for some prime p and $c \in \mathbb{N}$.

For a measurable subset E of K, let $|E| = \int_K \mathbf{1}_E(x)dx$, where $\mathbf{1}_E(x)$ is the characteristic function of E and dx is the Haar measure of K normalized so that $|\mathfrak{D}| = 1$. Then, it is easy to see that $|\mathfrak{P}| = q^{-1}$ and $|\mathfrak{p}| = q^{-1}$. We can decompose $\mathfrak{D}$ into q cosets of $\mathfrak{P}$. Thus $1 = |\mathfrak{D}| = q|\mathfrak{P}|$, this gives $|\mathfrak{P}| = q^{-1}$. Since $\mathfrak{P} = \mathfrak{p}\mathfrak{D}$, we have $|\mathfrak{p}| = q^{-1}$. It follows that if $x \neq 0$, and $x \in K$, then $|x| = q^k$ for some $k \in \mathbb{Z}$.

Let $\mathfrak{D}^\star = \mathfrak{D} \backslash \mathfrak{P} = \{x \in K : |x| = 1\}$. $\mathfrak{D}^\star$ is the group of units in $K^\star$. If $x \neq 0$, we can write $x = \mathfrak{p}^k x'$, with $x'\mathfrak{D}^\star$. Let $\mathfrak{P}^k = \mathfrak{p}^k\mathfrak{D} = \{x \in K : |x| \leq q^{-k}\}$,$k \in \mathbb{Z}$. These are called the *fractional ideals*. Each $\mathfrak{P}^k$ is compact and open and is a subgroup of K^+.

Also we have $\mathfrak{P}^{-k} = \mathfrak{p}^{-k}\mathfrak{D}$. So $|\mathfrak{P}^{-k}| = |\mathfrak{p}^{-k}||\mathfrak{D}| = q^k$. Summarizing everything above, we have the following theorem.

Theorem 1.4 (Summary)**.** *Let K be a local field. $\mathfrak{D} = \{x \in K : |x| \leq 1\}$, $\mathfrak{D}^\star = \{x \in K : |x| = 1\}$ and $\mathfrak{P} = \{x \in K : |x| < 1\}$. Then K is ultrametric, $\mathfrak{D}$ is the unique maximal compact subring in K, $\mathfrak{D}^\star$ is the group of units in $K^\star$ and $\mathfrak{P}$ is the unique maximal ideal in $\mathfrak{D}$. There is a $\mathfrak{p}$ in $\mathfrak{P}$ such that $\mathfrak{P} = \mathfrak{p}\mathfrak{D}$. The residue space $\mathfrak{I} = \mathfrak{D}/\mathfrak{P}$ is a finite field of characteristic p. If q is the number of elements in $\mathfrak{I}$, then the image of $K^\star$ in $(0, \infty)$ under the valuation $|.|$ is the subgroup of $(0, \infty)$ generated by q. $|\mathfrak{p}| = q^{-1}$. A Haar measure on $K^\star$ is given by $\frac{dx}{|x|}$.*

1.4. **Fourier analysis on local fields.**

If K is a local field, then there is a nontrivial, unitary, continuous character χ on K^+. It can be proved that K^+ is self dual. The existence of such a character follows from the Pontryagin duality theorem (Every topological group is canonically isomorphic to it's double dual). But in specific cases, it is easy to construct such characters directly.

Example 1.5.

(1) Let K be the 2-series field. Write $x = x_0 + \sum_{k=l}^{-1} a_k \mathfrak{p}^k$, $a_k = 0$ or 1, $x_0 \in \mathfrak{D}$. Define
$$\chi(\mathfrak{p}^k) = \begin{cases} 1, & \text{if } k = -1 \\ -1, & \text{if } k < -1 \end{cases}$$
and $\chi(x_0) = 1$. χ is a character on K^+ that is trivial on $\mathfrak{D}$ but is non-trivial on $\mathfrak{P}^{-1}$.

(2) Let K be the 2-adic field and write $x \in K$ as
$$x = x_0 + \sum_{k=l}^{-1} a_k 2^k,\ a_k = 0 \text{ or } 1,\ x_0 \in \mathfrak{D}$$
Define $\chi(2^k) = e^{2\pi i 2^k}$, $k \leq -1$ and $\chi(x_0) = 1$. χ is again a character on K^+ that is trivial on $\mathfrak{D}$, but is non-trivial on $\mathfrak{P}^{-1}$.

Let χ be a character that is trivial on $\mathfrak{D}$ but nontrivial on $\mathfrak{P}^{-1}$. We can find such a character by starting with any nontrivial character and rescaling. We will define such a character for a local field of positive characteristic. For $y \in K$, we define $\chi_y(x) = \chi(yx)$, $x \in K$.

We define the L^p spaces on K in the usual sense for $1 \leq p \leq \infty$. $\|f\|_p$ will denote the L^p norm. C_0 will denote the class of continuous functions on K that vanish at infinity. We endow C_0 with the L^∞ norm. We shall define convolution also in the usual way. If f and g are functions, then $h = f \star g$ is the function
$$h(x) = (f \star g)(x) = \int f(x-z)g(z)dz = \int f(z)g(x-z)dz$$

Definition 1.6 (Fourier transform for L^1 functions)**.** If $f \in L^1$, the fourier transform of f is the function $\widehat{f}$ defined by
$$\hat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}dx = \int_K f(x)\chi(-\xi x)dx$$

Just as in the classical Fourier analysis on $\mathbb{R}$, we can prove the following results.

Theorem 1.7. *Let $f \in L^1(K)$ and let $\widehat{f}$ be its Fourier transform.*

(1) *The map $f \to \widehat{f}$ is a bounded linear transformation of L^1 into L^∞, $\|\widehat{f}\|_\infty \leq \|f\|_1$.*
(2) *If $f \in L^1$ then $\widehat{f}$ is uniformly continuous.*
(3) *The translation operator τ_h is defined for $h \in K$ by $\tau_h f(x) = f(x-h)$ for any function f on K. If $f \in L^1$ then $\widehat{\tau_h f(x)} = \overline{\chi_h}\widehat{f}$ and $\widehat{\chi_h f} = \tau_h \widehat{f}$.*

(4) *The dilation operator* δ_k *is definied for* $k \in \mathbb{Z}$ *by* $(\delta_k f)(x) = f(\mathfrak{p}^k x)$. *If* $f \in L^1$ *then* $\widehat{(\delta_k f)} = q^k(\delta_{-k}\widehat{f})$
(5) *If* $f \in L^1(K) \cap L^2(K)$ *then* $\|\widehat{f}\|_2 = \|f\|_2$
(6) *(Riemann-Lebesgue) If* $f \in L^1$ *then* $\widehat{f}(x) \to 0$ *as* $|x| \to \infty$
(7) *The Fourier transform is unitary on* $L^2(K)$
(8) *If* $f \in L^p$, $1 \leq p \leq \infty$ *and* $g \in L^1$ *then* $f \star g \in L^p$ *and* $\|f \star g\|_p \leq \|f\|_p \|g\|_1$

In order to get some of the results above, we need to develop a bit of theory. To extend the Fourier transform to L^2 functions, we need the functions Φ_k. For $k \in \mathbb{Z}$, let Φ_k be the characteristic function for $\mathfrak{P}^k$. Also, a set of the form $h + \mathfrak{P}^k$ will be called a *sphere* with center h and radius q^{-k}. Note that if S and T are two spheres in K then S and T are either disjoint or one contains the other. Note that the characteristic function of $h + \mathfrak{P}^k = \tau_h \Phi_k = \Phi_k(. - h)$ and that this function is constant on cosets of $\mathfrak{P}^k$.

Definition 1.8 (Space of testing functions)**.** The set $\mathcal{S}$ of all finite linear combinations of functions of the form $\Phi_k(. - h)$ with $h \in K$ and $k \in \mathbb{Z}$

This class of functions can also be described in the following way. A function $g \in S$ if and only if there exist integers k, l such that g is constant on cosets of$\mathfrak{P}^k$ and is supported on $\mathfrak{P}^l$. It follows that $\mathcal{S}$ is closed under Fourier transform and is an algebra of continuous functions with compact support, which is dense in $C_0(K)$ as well as in $L^p(K)$, $1 \leq p < \infty$. We have the following theorem.

Theorem 1.9. *If* $g \in \mathcal{S}$ *is constant on cosets of* $\mathfrak{P}^k$ *and is supported on* $\mathfrak{P}^l$*, then* $\widehat{g} \in \mathcal{S}$ *is constant on cosets of* $\mathfrak{P}^{-l}$ *and is supported on* $\mathfrak{P}^{-k}$*.*

We will use the notation $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let χ_u be any character on K^+. Since $\mathfrak{D}$ is a subgroup of K^+, the restriction $\chi_u|_{\mathfrak{D}}$ is a character on $\mathfrak{D}$. Also, as characters on $\mathfrak{D}$,$\chi_u = \chi_v$ if and only if $u - v \in \mathfrak{D}$. That is,$\chi_u = \chi_v$ if $u + \mathfrak{D} = v + \mathfrak{D}$ and $\chi_u \neq \chi_v$ if $(u + \mathfrak{D}) \cap (v + \mathfrak{D}) = \phi$. To see this, we note that $\chi_u(x) = \chi_v(x)$ for all $x \in \mathfrak{D}$ iff $\chi((u - v)x) = 1$ for all $x \in \mathfrak{D}$ which is true iff χ is trivial on the sphere of radius $|u - v|$ which is true iff $u - v \in \mathfrak{D}$. Hence, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathfrak{D}$ in K^+, then $\{\chi_u(n) : n \in \mathbb{N}_0\}$ is a list of distinct characters on $\mathfrak{D}$. It can be seen that this list of characters is complete. For, if χ were a character on $\mathfrak{D}$ which is not on the list. Then extend it K (setting it to be zero outside $\mathfrak{D}$). Then $c_n = \int_K \chi(x)\overline{\chi_{u(n)}}(x)dx = 0$ for all n. Observe that $\chi \in L^1(K)$ and $\widehat{\chi}(\xi) = c_n$ if $\xi \in u(n) + \mathfrak{D}$. Thus $\widehat{\chi} = 0$ so $\chi = 0$ which is impossible since χ is a character on $\mathfrak{D}$.

Thus, we have the following proposition.

Theorem 1.10. *Let* $\{u(n) : n \in \mathbb{N}_0\}$ *be a complete list of (distinct) coset representatives of* $\mathfrak{D}$ *in* $K+$. *Then* $\{\chi_u(n) : n \in \mathbb{N}_0\}$ *is a complete list of (distinct) characters on* $\mathfrak{D}$*. Moreover, it is a complete orthonormal system on* $\mathfrak{D}$*.*

Given such a list of characters $\{u(n) : n \in \mathbb{N}_0\}$, we define the Fourier coefficients of $f \in L^1(\mathfrak{D})$ as

$$\widehat{f}(u(n)) = \int_{\mathfrak{D}} f(x)\overline{\chi_u(n)(x)}dx$$

.

The series $\sum_{n=0}^{\infty} \widehat{f}(u(n))\chi_u(n)(x)$ is called the *Fourier series* of f. From the standard L^2 theory for compact abelian groups we conclude that the Fourier series of f converges to f in $L^2(\mathfrak{D})$ and Parseval's identity holds :

$$\int_{\mathfrak{D}} |f(x)|^2 dx = \sum_{n=0}^{\infty} |\widehat{f}(u(n))|^2$$

Our ultimate aim in introducing the elements $u(n)$ is to get an analogue of integer translates in the real case in the general local field case. For that, we need to endow these elements $\{u(n) : n \in \mathbb{N}_0\}$ with an "natural" ordering. Note that $\mathfrak{I} = \mathfrak{D}/\mathfrak{P}$ is isomorphic to the finite field $\mathbb{F}_q$ and $\mathbb{F}_q$ is a c-dimensional vector space over $\mathbb{F}_p$. We choose a set $\{1 = \epsilon_0, \epsilon_1, \ldots, \epsilon_{c-1}\} \subset \mathfrak{D}^\star$ such that $\{\rho(\epsilon_0), \rho(\epsilon_1), \ldots, \rho(\epsilon_{c-1})\}$ is a basis for $\mathbb{F}_q$ over $\mathbb{F}_p$, where ρ is the projection map from $\mathfrak{D}$ to $\mathfrak{D}/\mathfrak{P}$. For $n \in \mathbb{N}_0$ such that $0 \leq n < q$, we have

$$n = a_0 + a_1 p + \ldots + a_{c-1} p^{c-1},\ 0 \leq a_k < p,\ k = 0, \ldots, c-1$$

Define, for $0 \leq n < q$, $u(n) = (a_0 + a_1\epsilon_1 + \ldots + a_{c-1}\epsilon_{c-1})\mathfrak{p}^{-1}$. Note that $u(0) = 0$, $\{u(n)\}_{n=0}^{q-1}$ is a complete set of coset representatives of $\mathfrak{D}$ in $\mathfrak{P}^{-1}$ and $|u(n)| = q$ if $1 \leq n < q$ (which follows from the corrolary of the ultrametric inequality). Now for $n \geq 0$, write

$$n = b_0 + b_1 q + \ldots + b_s q^s,\ 0 \leq b_k < q,$$

and set

$$u(n) = u(b_0) + \mathfrak{p}^{-1} u(b_1) + \ldots + \mathfrak{p}^{-s} u(b_s)$$

This defines $u(n)$ for $n \in \mathbb{N}_0$. It is important to note that in general, $u(m+n) \neq u(m) + u(n)$. However, it is true that for all $r, k \geq 0$,

$$u(rq^k) = \mathfrak{p}^{-k} u(r)$$

and for $r, k \geq 0$, $0 \leq t < q^k$,

$$u(rq^k + t) = u(rq^k) + u(t) = \mathfrak{p}^{-k} u(r) + u(t)$$

For brevity, we will write $\chi_n = \chi_{u(n)}$, $n \in \mathbb{N}_0$. As mentioned before, $\{\chi_n : n \in \mathbb{N}_0\}$ is a complete set of characters on $\mathfrak{D}$.

Let $\mathcal{U} = \{b_i\}_{i=0}^{q-1}$ be a fixed set of coset representatives of $\mathfrak{P}$ in $\mathfrak{D}$. Then every $x \in K$ can be expressed uniquely as

$$x = x_0 + \sum_{k=1}^{n} b_k \mathfrak{p}^k,\ x_0 \in \mathfrak{D},\ b_k \in \mathcal{U}$$

.

Let K be a local field of characteristic $p > 0$ and $\epsilon_0, \epsilon_1, ..., \epsilon_{c-1}$ be as above. We define a character χ on K as follows:

$$\chi(\epsilon_\mu \mathfrak{p}^{-j}) = \begin{cases} \exp(2\pi i/p) & \text{if } \mu = 0 \text{ and } j = 1 \\ 1, & \text{if } \mu = 1, \ldots, c-1 \text{ or } j \neq 1 \end{cases}$$

Note that χ is trivial on $\mathfrak{D}$ but nontrivial on $\mathfrak{P}^{-1}$. We have the following result for χ. We refer to [5].

Theorem 1.11. *For all $l, k \in \mathbb{N}_0$ we have $\chi(u(k)u(l)) = 1$.*

All the machinery above was developed in order to make sense of wavelets on local fields. Now we are fully equipped to look at how wavelets on local fields are constructed.

2. Constructing wavelets through Multiresolution Analysis (MRA).

In order to be able to define the concepts of MRA and wavelets on local fields, we need analogous notions of translation and dilation. Since $\cup_{j\in\mathbb{Z}}\mathfrak{p}^{-j}\mathfrak{D} = K$,we can regard $\mathfrak{p}^{-1}$ as the dilation (note that $|\mathfrak{p}^{-1}| = q$) and since $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representatives of $\mathfrak{D}$ in K, the set $\{u(n) : n \in \mathbb{N}_0\}$ can be treated as the translation set. We make the following definition. The following material is from [3].

Definition 2.1. A finite set $\{\psi_m : m = 1, 2, \ldots, M\} \subset L^2(K)$ is called a set of *basic wavelets* of $L^2(K)$ if the system $\{q^{j/2}\psi_m(\mathfrak{p}^{-j}. - u(k)) : 1 \leq m \leq M, j \in \mathbb{Z}, k \in \mathbb{N}_0\}$ forms an orthonormal basis for $L^2(K)$.

We compare the above definition to that of an orthonormal wavelet on R : A function $\psi \in L^2(\mathbb{R})$ such that $\psi_{j,k}(x) = 2^{j/2}\psi(2^jx - k)$ where $j, k \in \mathbb{Z}$ is an orthonormal basis for $L^2(\mathbb{R})$. But unlike the real case, it is not true that $u(k+l) = u(k) + u(l)$, which is one problem we did not face in the real case.

We now turn to the problem of defining an MRA on a local field. As mentioned before, local fields are of essentially two types. Connected local fields are either $\mathbb{R}$ and $\mathbb{C}$ and the concept of MRA on them is well understood. Totally disconnected local fields include the field $\mathbb{Q}_p$ which is of zero characteristic and the p-series field which is of finite characteristic. Wavelets on $L^2(\mathbb{Q}_p)$ have been studied by various authors. In this report, we give an outline of the definition of constructing an MRA on local fields of positive characteristic. At the end we shall give some concluding remarks on a possible approach to construct an MRA on Adele and Idele rings.

Definition 2.2 (MRA on local fields of positive characteristic)**.** Let K be a local field of characteristic $p > 0$, let $\mathfrak{p}$ be a prime element of K and $u(n) \in K$ for $n \in \mathbb{N}_0$, be as defined above. A multiresolution analysis (MRA) of $L^2(K)$ is a sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(K)$ satisfying the following properties:

(1) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;
(2) $\cup_{j\in\mathbb{Z}}V_j$ is dense in $L^2(K)$;
(3) $\cap_{j\in\mathbb{Z}}V_j = \{0\}$;
(4) $f \in Vj$ if and only if $f(\mathfrak{p}^{-1}.) \in V_{j+1}$ for all $j \in \mathbb{Z}$;
(5) There is a function $\phi \in V_0$, called the *scaling function*, such that $\{\phi(u(k)) : k \in \mathbb{N}_0\}$ forms an orthonormal basis for V_0.

The above conditions are not independent of each other, in the sense that condition (3) is a consequence of (1),(2),(4) and (5). Also, (2) follows from (1),(3),(4) and a slightly strengthened version of (5). Given an MRA $\{V_j : j \in \mathbb{Z}\}$, we define another sequence $\{W_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(K)$ by

$$W_j = V_{j+1} \ominus V_j.$$

That is, W_j is the orthogonal complement of V_j in V_{j+1}. These subspaces also satisfy

$$f \in W_j \text{ if and only if } f(\mathfrak{p}^{-1}.) \in W_{j+1}, j \in \mathbb{Z}.$$

Moreover they are mutually orthogonal and we have the following orthogonal decomposition.

$$L^2(K) = \bigoplus_{j\in\mathbb{Z}} W_j = V_0 \oplus \bigoplus_{j\geq 0} W_j$$

The dilation is induced by $\mathfrak{p}^{-1}$ and $|\mathfrak{p}^{-1}| = q$. As in the case of $\mathbb{R}^n$ we expect the existence of $q-1$ number of functions $\{\psi_1, \ldots, \psi_{q-1}\}$ to form a set of basic wavelets. From the above two properties of the W_j's it is clear that if $\{\psi_1, \ldots, \psi_{q-1}\}$ is a set of functions such that $\{\psi_m(. - u(k)) : 1 \leq m \leq q-1, k \in \mathbb{N}_0\}$ forms an orthonormal basis for W_0, then $\{q^{j/2}\psi_m(\mathfrak{p}^{-j}. - u(k)) : 1 \leq m \leq q-1, j \in \mathbb{Z}, k \in \mathbb{N}_0\}$ forms an orthonormal basis for $L^2(K)$.

For $f \in L^2(K)$, we define for $j \in \mathbb{Z}$ and $k \in \mathbb{N}_0$,

$$f_{j,k}(x) = q^{j/2} f(\mathfrak{p}^{-j}x - u(k))$$

Then it is easy to see that $\|f_{j,k}\|_2 = \|f\|_2$ and using the properties (3) and (4) of Theorem 2.7, that

$$\widehat{f_{j,k}}(\xi) = q^{-j/2}\overline{\chi_k(\mathfrak{p}^j\xi)}\widehat{f}(\mathfrak{p}^j\xi)$$

2.1. **Constructing a wavelet from an MRA.**

We provide a sketch of how to construct a wavelet from an MRA. Let $\{V_j : j \in \mathbb{Z}\}$ be an MRA of $L^2(K)$. Since $\phi \in V_0 \subset V_1$, and $\{\phi_{1,k} : k \in \mathbb{N}_0\}$ forms an orthonormal basis of V_1, there exists $\{\alpha_k^0 : k \in \mathbb{N}_0\} \in \ell^2(\mathbb{N}_0)$. such that

$$\phi(x) = \sum_{k \in \mathbb{N}_0} \alpha_k^0 \phi_{1,k}(x)$$

Taking Fourier transforms, we get

$$\widehat{\phi}(\xi) = q^{-1/2} \sum_{k \in \mathbb{N}_0} \alpha_k^0 \overline{\chi_k(\mathfrak{p}\xi)}\widehat{\phi}(\mathfrak{p}\xi) = m_0(\mathfrak{p}\xi)\widehat{\phi}(\mathfrak{p}\xi)$$

Where $m_0(\xi) = q^{-1/2}\sum_{k \in \mathbb{N}_0} \alpha_k^0 \overline{\chi_k(\xi)}$

Definition 2.3. A function f on K is called *integral-periodic* if

$$f(x + u(k)) = f(x) \text{ for all : } k \in \mathbb{N}_0$$

It has been proven in [5] that the function m_0 is integral-periodic and is in $L^2(\mathfrak{D})$.

As in the case for $\mathbb{R}^n$, it can be shown that if we can find integral-periodic functions m_i for $1 \leq i \leq q-1$, such that the matrix

$$M(\xi) = \Big[m_i(\xi + \mathfrak{p}u(j)) \Big]_{i,j=0}^{q-1}$$

is unitary for a.e. $\xi \in \mathfrak{D}$, then $\{\psi^1, \ldots, \psi^{q-1}\}$ forms a set of basic wavelets for $L^2(K)$, where

$$\widehat{\psi^i}(\xi) = m_i(\mathfrak{p}\xi)\widehat{\phi}(\mathfrak{p}\xi)$$

In other words, if

$$m_i(\xi) = q^{-1/2} \sum_{n \in \mathbb{N}_0} \alpha_k^i \overline{\chi_k(\xi)}$$

where $\{\alpha_k^i : k \in \mathbb{N}_0\} \in \ell^2(\mathbb{N}_0)$, then,

$$\psi^i(x) = q^{1/2} \sum_{k \in \mathbb{N}_0} \alpha_k^i \phi(\mathfrak{p}^{-1}x - u(k))$$

Example 2.4 (Haar Wavelets)**.** Let $\phi = \mathbf{1}_{\mathfrak{D}}$. Define $V_j = \overline{\text{span}}\{\phi(\mathfrak{p}^{-1}x - u(k)) : k \in \mathbb{N}_0\}$. Then $\{V_j : j \in \mathbb{Z}\}$ forms an MRA of $L^2(K)$. This is called the Haar MRA. It can now be shown that $\{\psi^i : 1 \leq i \leq q-1\}$ is a set of basic wavelets of $L^2(K)$ where

$$\psi^i(x) = \sum_{j=0}^{q-1} a_{ij} q^{1/2} \phi(\mathfrak{p}^{-1}x - u(j)) \; 1 \leq j \leq q-1$$

where $A = (a_{ij})_{i,j=0}^{q-1}$ is an arbitary unitary matrix such that $a_{0j} = q^{-1/2}$ for $0 \leq j \leq q-1$. It can be shown that the columns of the corresponding matrix $M(\xi)$ form an orthonormal basis for $\mathbb{C}^q$ hence the matrix is unitary a.e.

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