ON CERTAIN GROUP THEORETIC ASPECTS OF THE FOURIER TRANSFORM

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Abstract. Harmonic analysis has at it's heart the ubiquitous Fourier Transform, which has many widespread applications and generalizations across various areas of pure and applied mathematics. On the real line $\mathbb{R}$, it is given as follows.

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\xi} f(x)dx$$

In this project report, we dissect the above definition of the Fourier Transform and see how it generalizes to other structures. Indeed, $\mathbb{R}$ is a locally compact topological group; the measure used is the translation invariant Lebesgue measure which is the Haar measure on $\mathbb{R}$; the functions $e^{-2\pi i x\xi}$ out of which $\mathcal{F}$ is fashioned are its irreducible representations; and $\mathcal{F}$ gives the Gelfand transform on $L^1(\mathbb{R})$, and the decomposition of the regular representation of $\mathbb{R}$ into its irreducible components. In this project, we analyze each of the statements above in detail and see exactly how the abstraction of the Fourier Transform takes place. In the process, we shall see the development of Representation Theory of locally compact groups, along with other results of Harmonic Analysis..

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1. Introduction

The Fourier transform on $\mathbb{R}$ is given by

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\xi} f(x)dx$$

In the following pages, we proceed with a dissection of the definition of the Fourier Transform given above, and see how it can be generalized. For sake of clarity, let us list the various aspects of the definition.

- $\mathbb{R}$ is a **locally compact topological group**.
- dx, the Lebesgue measure on $\mathbb{R}$ is a **Haar measure**.
- The functions $e^{-2\pi i x\xi}$ are the **irreducible representations** of $\mathbb{R}$.
- The transform $\mathcal{F}$ is the **Gelfand Transform** of the Banach Algebra $L^1(\mathbb{R})$.
- The transform $\mathcal{F}$ gives the decomposition of the **regular representation** of $\mathbb{R}$ into its irreducible components.

We shall now consider each of the above in detail.

2. Locally compact topological groups.

Definition 2.1. A **topological group** is a group G equipped with a topology with respect to which the group operations are continuous; that is, $(x, y) \to xy$ is continuous from $G \times G$ to G and $x \to x^{-1}$ is continuous from G to G.

Here are some basic general properties of topological groups.

Theorem 2.2. *Let G be a topological group.*

(1) *The topology of G is invariant under translations and inversion; that is, if U is open then so are xU, Ux, and U^{-1} for any $x \in G$.*
(2) *For every neighborhood U of 1 (the identity) there is a symmetric neighborhood V of 1 such that $VV \subseteq U$. (A symmetric neighborhood A is one in which is closed under taking inverses.)*
(3) *If H is a subgroup of G, then so is $\overline{H}$.*
(4) *Every open subgroup of G is closed.*
(5) *If A and B are compact sets in G, so is AB.*
(6) *Suppose H is a subgroup of G. If H is closed, G/H is Hausdorff.*
(7) *If G is locally compact, so is G/H.*
(8) *If H is normal, G/H is a topological group.*
(9) *If G is T_1 then G is Hausdorff. If G is not T_1 then $\overline{\{1\}}$ is a closed normal subgroup, and $G/\overline{\{1\}}$ is a Hausdorff topological group.*

From (9) above we note that it is essentially no restriction to assume that G is Hausdorff, for if not, we simply work with $G/\overline{\{1\}}$. So we assume G is a Hausdorff topological group from now. Now, if G is also locally compact with respect to its topology, then we call G a **locally compact Hausdorff topological group**.

2.1. **Examples of locally compact groups.** Some examples include the additive group $\mathbb{R}^n$, closed subgroups of $GL_n(\mathbb{R})$ and the group $\mathbb{T}$ of all complex numbers of modulus 1.
Also, the group $\mathbb{Z}_2^\omega$ of countable copies of the group $\mathbb{Z}_2$ and the group $\mathbb{Q}_p$ of p-adic numbers also constitute examples.

3. Haar measures on locally compact groups

Definition 3.1. A left (resp. right) Haar measure on G is a nonzero Radon(i.e., inner regular and locally finite) measure μ on G that satisfies $\mu(xE) = \mu(E)$ (resp. $\mu(Ex) = \mu(E)$) for every Borel set $E \subset G$ and every $x \in G$.

Let $C_c(G)$ denote the set of all compactly supported functions on G. We set

$$C_c^+(G) = \{f \in G : f \geq 0, f \not\equiv 0\}$$

Theorem 3.2. *Let μ be a Radon measure on the locally compact group G, and let $\widetilde{\mu}(E) = \mu(E^{-1})$.*

(1) *μ is a left Haar measure if and only if $\widetilde{\mu}$ is a right Haar measure*
(2) *μ is a left Haar measure if and only if $\int L_y(f)d\mu = \int fd\mu$ for every $f \in C_c^+(G)$ and every $y \in G$. Where $L_y f(x) = f(y^{-1}x)$.*

Proof. (1) is obvious. For (2), for any Radon measure μ we have $\int L_y(f)d\mu = \int fd\mu_y$ where $\mu_y(E) = \mu(yE)$, as one sees by approximating f by simple functions. So given this fact, if we assume μ is a left Haar measure, we have $\int L_y(f)d\mu = \int fd\mu$. If we assume $\int L_y(f)d\mu = \int fd\mu$ holds for all $f \in C_c^+(G)$, we have that it holds for all $f \in C_c(G)$ as $C_c(G)$ is the linear span of $C_c^+(G)$. So one has $\mu = \mu_y$ by the uniqueness part of the Riesz Representation Theorem. □

So the above theorem tells us that choosing to study either left or right Haar measures is basically the same thing. The more common choice is to study left Haar measures, and this is what we adhere to. The next natural question one can ask is whether Haar measures exist for every group, and if they are unique. Both these questions are answered in the affirmative by the following theorems.

Theorem 3.3. *Every locally compact group G possesses a left Haar measure λ.*

One also has the following two theorems.

Theorem 3.4. *If λ is a left Haar measure on G, then $\lambda(U) > 0$ for every nonempty open set U and $\int f d\lambda > 0$ for every $f \in C_c^+(G)$.*

Theorem 3.5. *If λ and μ are two left Haar measures on a group G then there exists a constant $c > 0$ such that $\mu = c\lambda$.*

3.1. **Explicit examples of Haar measures on groups.** We state a general result which covers quite a few cases of interest.

Theorem 3.6. *Suppose G is a topological group with a smooth differentiable structure, i.e., a Lie group, and say the underlying manifold of G is an open subset of $\mathbb{R}^N$. Say the left translations are given by affine maps: $xy = A(x)y + b(x)$, where $A(x)$ is a linear transformation on $\mathbb{R}^N$ and $b(x) \in \mathbb{R}^N$. Then $|detA(x)|^{-1}dx$ is a left Haar measure on G, where dx denotes the Lebesgue measure on $\mathbb{R}^N$.*

As applications of the above result, we have the following.

(1) $dx/|x|$ is a Haar measure on the multiplicative group $\mathbb{R}\backslash\{0\}$.
(2) $dx\,dy/(x^2 + y^2)$ is a Haar measure on the multiplicative group $\mathbb{C}\backslash\{0\}$.
(3) The Lebesgue measure $\prod_{i<j} d\alpha_{ij}$ is a left and right Haar measure on the group of $n \times n$ real matrices (α_{ij}) such that $\alpha_{ij} = 0$ for $i > j$ and $\alpha_{ii} = 1$ for $1 \leq i \leq n$.
(4) $|\det T|^{-n} dT$ is a left and right Haar measure on the group $GL_n(\mathbb{R})$ of invertible linear transformations of $\mathbb{R}^n$, where dT is the Lebesgue measure on the vector space of all real $n \times n$ real matrices.
(5) The $ax + b$ **group** G is the group of all affine transformations $x \to ax + b$ with $a > 0$ and $b \in \mathbb{R}$. On G, $da\,db/a^2$ is a left Haar measure and $da\,db/a$ is a right Haar measure.
(6) If $G_1, \ldots, G_n$ are locally compact groups with left Haar measures $\lambda_1, ..., \lambda_n$, then left Haar measure on $G = \prod_j G_j$ is obviously the Radon product of $\lambda_1, \ldots, \lambda_n$, that is, the Radon measure on G associated to the linear functional
$$f \to \int \cdots \int f(x_1, \ldots, x_n) d\lambda_1(x_1) \ldots d\lambda_n(x_n)$$
. In case of compact groups, one can also allow infinitely many factors, provided that one normalizes the Haar measures to have total mass 1.
(7) On the p-adic numbers $\mathbb{Q}_p$, one has the Haar measure
$$\lambda(E) = \inf \left\{ \sum_{j=1}^{\infty} p^{m_j} : E \subset \bigcup_{1}^{\infty} \overline{B}(p^{m_j}, x_j) \right\}$$
Where $B(x, r)$ is the ball with center x and radius r.

3.2. **The Modular function.** The modular function measures how far the Haar measure λ on a group G fails to be right-invariant. If, for $x \in G$ we define $\lambda_x(E) = \lambda(Ex)$, then λ_x is again a Haar measure. So by the uniqueness theorem there is a unique number $\Delta(x) > 0$ such that $\lambda_x = \Delta(x)\lambda$. This function $\Delta : G \to (0, \infty)$ is called the **modular function** of the group G.

We have the following result about Δ.

Theorem 3.7. *Δ is a continuous homomorphism from G to $\mathbb{R}^\times$, the multiplicative group of nonzero reals. Furthermore, we have*

$$\int R_y(f) d\lambda = \Delta(y^{-1}) \int f d\lambda$$

Where $R_y f(x) = f(xy)$.

The above theorem can also be stated as follows, once we make the identification $d\lambda(x) \to \int f d\lambda(x)$.

$$d\lambda(xy^{-1}) = \Delta(y^{-1}) d\lambda(x)$$

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Given a left Haar measure λ, we have a right Haar measure ρ defined as $\rho(E) = \lambda(E^{-1})$. These two are related as follows.

Theorem 3.8. *If ρ and λ are defined as above, then we have*

$$d\rho(x) = \Delta(x^{-1}) d\lambda(x)$$

The above formula can be restated as

$$d\lambda(x^{-1}) = \Delta(x^{-1}) d\lambda(x)$$

Definition 3.9. A group G is called **unimodular** if $\Delta \equiv 1$, that is, a left Haar measure is also a right Haar measure. Unimodularity is a highly desirable property for a group to have. Obviously, Abelian and discrete groups are unimodular, but there are other classes of groups which are also unimodular.

- If K is a compact subgroup of a group G, then $\Delta|_K \equiv 1$. This shows that if G was a compact group, then G is unimodular.
- If $G/[G, G]$ is compact, then G is unimodular.

3.3. **L^p spaces, Convolutions and Approximate Identities.** On the group G and for $1 \leq p \leq \infty$ we define $L^p(G)$ to be the usual L^p space with respect to the Haar measure λ on G, and with the usual L^p norm. Define the involution $f^\star(x) = \Delta(x^{-1})\overline{f(x^{-1})}$.
Now, for $f, g \in L^1(G)$, define

$$f \star g(x) = \int f(y) g(y^{-1}x) dy$$

By Fubini's theorem, it is easy to see that the above integral is absolutely convergent for almost every x and that $\|f \star g\|_1 \leq \|f\|_1 \|g\|_1$. With the convolution product and the involution, we see that $L^1(G)$ becomes a Banach $\star$-algebra.

Convolution can be extended to other L^p spaces as well.

Theorem 3.10. *Suppose $1 \leq p \leq \infty$, $f \in L^1$ and $g \in L^p$.*

1. *We have $f \star g \in L^p$ and $\|f \star g\|_p \leq \|f\|_1 \|g\|_p$.*
2. *If G is not unimodular, we have $g \star f \in L^p$ when f has compact support.*
3. *If G is unimodular, then (1) holds with $f \star g$ replaced by $g \star f$ also.*

(4) *When $p = \infty$, $f \star g$ is continuous, and under the conditions of (2) or (3), so is $g \star f$.*
(5) *Suppose G is unimodular. If $f \in L^p(G)$ and $g \in L^q(G)$ where $1 < p, q < \infty$ and $p^{-1} + q^{-1} = 1$, then $f \star g \in C_0(G)$ and $\|f \star g\|_{sup} \leq \|f\|_p \|g\|_q$*
(6) *If G is unimodular, we have $L^p \star L^q \subset L^r$ and that $\|f \star g\|_r \leq \|f\|_p \|g\|_q$ whenever $p^{-1} + q^{-1} = r^{-1} + 1$.*

The last result (6) follows from the Riesz-Thorin interpolation theorem.

Suppose now that G is a discrete group and δ is the function defined by $\delta(e) = 1$ and 0 otherwise. Then we can see that $f \star \delta = \delta \star f = f$ for all $f \in L^1(G)$. When G is not discrete, then there is no function with this property. However, we can always find a net of functions (f_α) such that for every $g \in L^1(G)$, $f_\alpha \star g \to g$ and $g_\alpha \star f \to g$ in $L^1(G)$. Such an (f_α) is called an **approximate identity**. The following theorem gives a construction of an approximate identity.

Theorem 3.11. *Let $\mathcal{N}$ be a neighbourhood base at e consisting of compact symmetric neighbourhoods. For each $U \in \mathcal{N}$,let $f_U = \lambda(U)^{-1}\chi_U$. Then consider the net (f_U). For every $g \in L^p(G)$, $1 \leq p < \infty$, we have $f_U \star g \to g$ and $g \star f_U \to g$ in $L^p(G)$.*

4. Gelfand transforms.

In this section we digress a little and show how the Gelfand transform is a generalization of the Fourier transform in the familiar case of the Banach algebra $L^1(\mathbb{Z})$.

Definition 4.1. Let $\mathcal{A}$ be a Banach algebra.

- A linear functional ϕ on $\mathcal{A}$ is called **multiplicative** if ϕ is non-trivial and $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in \mathcal{A}$
- For a Banach algebra $\mathcal{A}$, we denote $\mathcal{M}_\mathcal{A}$ to be the set of all multiplicative linear functionals on $\mathcal{A}$. $\mathcal{M}_\mathcal{A}$ becomes a compact Hausdorff space when it inherits the weak* topology of the dual space of $\mathcal{A}$. $\mathcal{M}_\mathcal{A}$ is called the **maximal ideal space** of $\mathcal{A}$.
- Let $\mathcal{A}$ be a Banach algebra. Then the mapping $\Gamma : \mathcal{A} \to \mathbf{C}(\mathcal{M}_\mathcal{A})$ defined by
$$\Gamma(x)(\phi) = \phi(x)$$
is called the **Gelfand transform** of $\mathcal{A}$.

It is true that the Gelfand transform of the Banach algebra $L^1(G)$ where G is a locally compact commutative group is nothing but the Fourier transform on G (The meaning of these statements will be made clear later). We shall prove this fact for the Banach algebra $L^1(\mathbb{Z})$. The commutative Banach algebra $L^1(\mathbb{Z})$ has norm
$$\|f\| = \sum_{n=-\infty}^{\infty} |f(n)|$$
and the product is given by convolution
$$f * g(n) = \sum_{k=-\infty}^{\infty} f(n-k)g(k)$$

Theorem 4.2. *The maximal ideal space of $L^1(\mathbb{Z})$ is homeomorphic to the unit circle $\mathbf{S}^1$.*

Proof. For each $z \in \mathbf{S}^1$, define $\phi_z : L^1(\mathbb{Z}) \to \mathbb{C}$ by

$$\phi_z(f) = \sum_{n=-\infty}^{\infty} f(n) z^n$$

The functional ϕ_z is clearly bounded, linear and non-trivial. Moreover, a simple application of Fubini's theorem shows us that $\phi_z(f*g) = \phi_z(f)\phi_z(g)$. So ϕ_z belongs to the maximal ideal space of $L^1(\mathbb{Z})$ Define $\Phi : \mathbf{S}^1 \to \mathcal{M}_{L^1(\mathbb{Z})}$ by $\Phi(z) = \phi_z$. We show that Φ is a surjective homeomorphism.

For each $n \in \mathbb{Z}$ let f_n be the function in $L^1(\mathbb{Z})$ such that $f_n(n) = 1$ and 0 otherwise. It is easy to check that f_0 is the identity of the Banach algebra $L^1(\mathbb{Z})$, $\|f_n\| = 1$ for all n, and $f_n * f_k = f_{n+k}$ for all $n, k \in \mathbb{Z}$.

Φ is one-one : If we had $\Phi(z_1) = \Phi(z_2)$, that is $\phi_{z_1} \equiv \phi_{z_2}$, then in particular, we have $\phi_{z_1}(f_1) = \phi_{z_2}(f_1)$, which shows that $z_1 = z_2$.

Φ is onto : Let $\phi \in \mathcal{M}_{L^1(\mathbb{Z})}$ and let $z_0 = \phi(f_1)$. Clearly $|z_0| \leq 1$. On the other hand, $\mathbf{1} = f_1 * f_{-1}$ implies that $1 = z_0\phi(f_{-1})$ and hence $1 = |z_0||\phi(f_{-1})| \leq |z_0|$. So $z_0 \in \mathbf{S}^1$. Since ϕ is multiplicative,

$$\phi(f_n) = \phi(f_1 * f_1 * \cdots * f_1) = \phi(f_1)^n = z_0^n$$

for all positive integers n. It follows from this and the identity $\mathbf{1} = f_n * f_{-n}$ that $\phi(f_n) = z_0^n$ for all integers n. If $f \in L^1(\mathbb{Z})$, then we can write

$$f = \sum_{n=-\infty}^{\infty} f(n) f_n$$

with the series converging in $L^1(\mathbb{Z})$. By continuity and linearity of ϕ, we have

$$\phi(f) = \sum_{n=-\infty}^{\infty} f(n)\phi(f_n) = \sum_{n=-\infty}^{\infty} f(n) z_0^n = \phi_{z_0}(f)$$

for all f in $L^1(\mathbb{Z})$ and so Φ is onto.

If $z_\alpha \to z$ in $\mathbf{S}^1$, then by an easy $\epsilon - \delta$ argument, we get

$$\lim_\alpha \sum_{n=-\infty}^{\infty} f(n) z_\alpha^n = \sum_{n=-\infty}^{\infty} f(n) z^n$$

for every $f \in L^1(\mathbb{Z})$ and hence $\Phi(z_\alpha) \to \Phi(z)$, showing that Φ is continuous. Again since both $\mathbf{S}^1$ and $\mathcal{M}_{L^1(\mathbb{Z})}$ are compact Hausdorff, we get that Φ is a homeomorphism. □

If we identify the maximal ideal space of $L^1(\mathbb{Z})$ with $\mathbf{S}^1$, then the Gelfand transform $\Gamma : L^1(\mathbb{Z}) \to \mathbf{C}(\mathbf{S}^1)$ takes the following form.

$$\Gamma(f)(z) = \sum_{n=-\infty}^{\infty} f(n) z^n$$

where $z \in \mathbf{S}^1$. So z can be written as $z = e^{-it}$ for some $t \in \mathbb{R}$. So the formula becomes

$$\Gamma(f)(t) = \sum_{n=-\infty}^{\infty} f(n) e^{-int} \tag{4.1}$$

This transformation takes $f \in L^1(\mathbb{Z})$ to $\mathbf{C}(\mathbf{S}^1)$.

For the general case, where $\mathbb{Z}$ is replaced by a locally compact Abelian group G, one needs an appropriate definition of what the Fourier transform means for an arbitary locally compact Abelian group. For this, we shall need the notion of the "dual group" of a locally compact Abelian group G. For this, we delve into representation theory.

5. Representations on locally compact groups.

Looking at the proof given in the last section we notice that the functions e^{int} appearing and these seem to be connected to the group $\mathbf{S}^1$, (Explicitly, we have that the dual group of $\mathbb{Z}$ is the group $\mathbf{S}^1$) and if we need to generalize the above proof to an arbitary Abelian group G, then it is important that we find similar analogues of the functions above. In order to solve this problem, we need to develop a little bit of Representation theory.

Definition 5.1. Let G be a locally compact group. A **unitary representation** of G is a homomorphism π from G into the group $\mathcal{U}(H_\pi)$ of unitary operators on some nonzero Hilbert space H_π that is continuous with respect to the strong operator topology.

So we have $\pi(xy) = \pi(x)\pi(y)$ and $\pi(x^{-1}) = \pi(x)^{-1} = \pi(x)^*$ and for which $x \to \pi(x)u$ is continuous from G to H_π for any $u \in H_\pi$. H_π is called the **representation space** of π and its dimension is called the **degree** of π.
So a representation assigns to each element $x \in G$ a unitary operator $\pi(x) \in \mathcal{L}(H_\pi)$.

5.1. Examples of Representations.

(1) (a) $G = \mathbf{S}^1$ and $H = \mathbb{C}$.
For each $n \in \mathbb{Z}$, define $\pi_n(e^{i\theta})z = e^{in\theta}z$ for all $e^{i\theta} \in \mathbb{S}^1$, $z \in \mathbb{C}$
(b) $G = \mathbb{R}$ and $H = \mathbb{C}$.
For each $x \in \mathbb{R}$ define $\pi_x(y)z = e^{ixy}z$ for all $y \in \mathbb{R}$, $z \in \mathbb{C}$.
These are the unitary representations of $\mathbf{S}^1$ and $\mathbb{R}$ respectively.

(2) $G = GL_n(\mathbb{R})$ and $H = \mathbb{C}^n$. Define $\pi(A)z = Az$ for all $A \in G$ and $z \in \mathbb{C}^n$. Then π is a representation which is not unitary.

(3) $G = U(n)$, the unitary group of matrices over C and $H = \mathbb{C}^n$. Then $\pi(A)z = Az$ for all $A \in U(n)$ and $z \in \mathbb{C}^n$ is a unitary representation.

(4) Let G be a locally compact group and $H = L^2(G)$. Define

$$(L(g)f)(s) = f(g^{-1}s) \text{ for all } g \in G \text{ and } f \in H$$

Then $g \to L(g)$ is called the **left regular representation** of G, and it is unitary because it is surjective and because of the invariance of the Haar measure, we get $\langle L(g)(f_1), L(g)(f_2)\rangle = \langle f, g\rangle$, where $\langle ., .\rangle$ is the inner product on the Hilbert space $L^2(G)$. Also the continuity of $g \to L(g)f$ for a fixed f can be proven easily.
Similarly, define

$$(R(g)f)(s) = f(sg), g \in G, f \in H$$

Exactly as above, $g \to R(g)$ is a unitary representation on G called the **right regular representation**.

We need a few more definitions.

Definition 5.2. Let G be a locally compact group.

(1) Let π_1 and π_2 be two representations on the group G. An **intertwining operator** for π_1 and π_2 is a bounded linear map $T : H_{\pi_1} \to H_{\pi_2}$ such that $T\pi_1(x) = \pi_2(x)T$ for all $x \in G$. The set of all intertwining operators is denoted $C(\pi_1, \pi_2)$. If $C(\pi_1, \pi_2)$ contains a unitary operator, then we say that π_1 and π_2 are **unitarily equivalent**.
(2) For a representation π, denote let $C(\pi) := C(\pi, \pi)$. $C(\pi)$ is called the **centralizer** or **communtant** of π.
(3) Let $M \subset H_\pi$ be a closed subspace. We say M is an invariant subspace of the representation π if we have $\pi(x)M \subset M$ for all $x \in G$. If $M \neq \{0\}$ is invariant, then the restriction of π to M, $\pi^M(x)$ defines a representation of G on M, called a **subrepresentation** of π
(4) If $\{\pi_i\}_{i\in I}$ is a family of unitary representations, their **direct sum** $\oplus\pi_i$ is the representation π on $H = \bigoplus H_{\pi_i}$, defined by $\pi(x)(\sum v_i) = \sum \pi_i(x)v_i$ $(v_i \in H_{\pi_i}$.
(5) If π admits an invariant subspace that is nontrivial, then π is called **reducible**, otherwise π is **irreducible**.
(6) If π is a unitary representation of G and $u \in H_\pi$, the closed linear span M_u of $\{\pi(x)u : x \in G\}$ is called the **cyclic subspace** generated by u. Clearly M_u is invariant under π. If $M_u = H_\pi$, u is called a **cyclic vector** for π. π is called a **cyclic representation** if it has a cyclic vector.

One can see that all one-dimensional representations are irreducible, for the only subspaces of a one-dimensional Hilbert space are $\{0\}$ and itself. We will see that if G is a locally compact Abelian topological group, then all of its irreducible representations are one-dimensional. To prove this, we shall require the following fundamental lemma.

Lemma 5.3 (Schur's Lemma)**.** *Let G be a locally compact topological group.*

(1) *A unitary representation π of G is irreducible if and only if $C(\pi)$ contains scalar multiples of identity.*
(2) *Suppose π_1 and π_2 are irreducible unitary representations of G. If π_1 and π_2 are unitarily equivalent then $C(\pi_1, \pi_2)$ is one dimensional, otherwise, $C(\pi_1, \pi_2) = \{0\}$*

Theorem 5.4. *If G is a locally compact Abelian topological group, then all of its irreducible representations are one dimensional.*

Proof. If π is a representation of G, since G is Abelian the operators $\pi(x)$ all commute with one another. So $\pi(x) \in C(\pi)$ for all $x \in G$. So now if π was irreducible, by Schur's lemma, we have that $C(\pi)$ contains only scalar multiples of the identity. So for every $x \in G$, there is a $\lambda_x \in \mathbb{C}$ such that $(\pi(x))u = \lambda_x u$. So any one-dimensional subspace of H_π is invariant under π. So we must have $\dim H_\pi = 1$, else π would not be irreducible. □

The irreducible unitary representations of a locally compact group G are the basic building blocks of the harmonic analysis associated to G. Ofcourse, we need to ensure that irreducible unitary representations always exist, and exactly this is assured by the famous Gelfand-Raikov theorem.

Theorem 5.5 (Gelfand-Raikov theorem)**.** *If G is any locally compact group, the irreducible unitary representations of G separate points on G. That is, if x and y are distinct points of G, there is an irreducible representation π such that $\pi(x) \neq \pi(y)$.*

Once this is guaranteed, the major questions of harmonic analysis are the following.

(1) Describe all the irreducible unitary representations of G, up to unitary equivalence.
(2) Determine how arbitrary unitary representations of G can be built out of irreducible ones.
(3) Given a specific unitary representation of G such as the regular representation, show concretely how to build it out of irreducible ones.

Though a serious study of the above three questions is out of the scope of our project, we make a few general remarks on each of them.
The answer to question (1) shall depend strongly on the nature of the group in question. There is a general technique that can be used to classify irreducible representations of many groups, via induced representations.
As to question (2), one might hope that every unitary representation is the direct sum of irreducible subrepresentations, which is the case when G is a compact group, but unfortunately, this is not true in general. What is true however, is that every unitary representation is a direct integral of irreducible representations, via the Fourier inversion formula. Uniqueness of this decomposition is quite a delicate issue.
For question (3), if we consider the regular representation, the answer is what is called the "Plancherel theorem" for the group G.

Resuming our study on locally compact Abelian groups, it follows from theorem 5.4 that if π is an irreducible unitary representation of an Abelian group G, then we can take $H_\pi = \mathbb{C}$, and thus we shall have $\pi(g) = \lambda_g I$ for some $\lambda_g \in \mathbf{S}^1$. So the map $g \to \lambda_g$ is a continuous homomorphism from G to $\mathbf{S}^1$. Such a homomorphism is called a **character** of G.

Definition 5.6. Let G be a locally compact Abelian group. The set of all characters of G is denoted $\widehat{G}$, and is called the **dual group** of G. In general, for a non-abelian group G, $\widehat{G}$ is used to denote the set of equivalence classes of irreducible unitary representations of G. Because all irreducible unitary representations of an Abelian group G are one-dimensional, we can identify them as characters. So the two notations are consistent.

We list a few results regarding $\widehat{G}$. For reasons of symmetry, we shall write $\xi(x) = \langle x, \xi \rangle$ for $\xi \in \widehat{G}$.

Theorem 5.7. *Let G be a locally compact Abelian group and let $\widehat{G}$ be its dual group.*

(1) *We can identify $\widehat{G}$ with the set of all multiplicative linear functionals on $L^1(G)$ via the identification*

$$\xi \longrightarrow \xi(f) = \int \xi(x) f(x) dx \tag{5.1}$$

Conversely, every multiplicative linear functional on $L^1(G)$ is given by integration against a character.

(2) $\widehat{G}$ *is clearly an Abelian group under pointwise multiplication; it's identity is the constant function* ***1*** *and*

$$\langle x, \xi^{-1} \rangle = \langle x^{-1}, \xi \rangle = \overline{\langle x, \xi \rangle}$$

(3) *The topology of compact convergence on* $\widehat{G}$*, which makes its group operations continuous, coincides with the weak* topology which* $\widehat{G}$ *inherits as a subset of* L^∞*. (Recall that we identify* $\widehat{G}$ *to be the maximal ideal space of* $L^1(G)$*, which are functionals, and so contained in the dual of* $L^1(G)$*, which is contained in* $L^\infty(G)$*.)*

(4) $\widehat{G} \cup \{0\}$ *is weak* compact. So* $\widehat{G}$ *is a locally compact Abelian group.*

(5) *If* G *is compact and the Haar measure is normalized so that* $|G| = 1$*, then we have that* $\widehat{G}$ *is an orthonormal set in* $L^2(G)$

(6) *If* G *is discrete then* $\widehat{G}$ *is compact. If* G *is compact then* $\widehat{G}$ *is discrete.*

5.2. **Examples of the dual group.**

Theorem 5.8. *Here are some of the examples of dual groups.*

(1) $\widehat{\mathbb{R}} \cong \mathbb{R}$ *with the pairing* $\langle x, \xi \rangle = e^{2\pi i \xi x}$.

(2) $\widehat{\boldsymbol{S}^1} \cong \mathbb{Z}$ *with the pairing* $\langle \alpha, n \rangle = \alpha^n$.

(3) $\widehat{\mathbb{Z}} \cong \boldsymbol{S}^1$ *with the pairing* $\langle n, \alpha \rangle = \alpha^n$.

(4) *If* Z_k *is the additive group of integers mod* k*, then* $\widehat{Z_k} \cong Z_k$ *with the pairing* $\langle m, n \rangle = e^{2\pi i mn/k}$.

(5) *If* $G_1, \ldots, G_n$ *are locally compact Abelian groups, then*

$$(G_1 \widehat{\times \cdots \times} G_k) \cong \widehat{G_1} \times \cdots \times \widehat{G_k}$$

(6) *From the above, we have* $\widehat{\mathbb{R}^n} \cong \mathbb{R}^n$, $\widehat{\mathbb{Z}^n} \cong (\boldsymbol{S}^1)^n$, $\widehat{(\boldsymbol{S}^1)^n} \cong \mathbb{Z}^n$, $\widehat{G} = G$ *where* G *is a finite abelian group.*

Proof. (1) If $\phi \in \widehat{R}$, then we have $\phi(0) = 1$, so there exists an $a > 0$ such that $A = \int_0^a \phi(t)dt \neq 0$. We now have

$$A\phi(x) = \int_0^a \phi(x+t)dt = \int_x^{a+x} \phi(t)dt$$

So ϕ is differentiable and we have

$$\phi'(x) = A^{-1}[\phi(a+x) - \phi(x)] = c\phi(x)$$

where $c = A^{-1}[\phi(a) - 1]$. It follows that $\phi(t) = e^{ct}$, and since $|\phi| = 1$, we get $c = 2\pi i \xi$ for some $\xi \in \mathbb{R}$

(2) Since we have $\mathbf{S}^1 \cong \mathbb{R}/\mathbb{Z}$, via the identification of $x \in \mathbb{R}/\mathbb{Z}$ with $\alpha = e^{2\pi i x} \in \mathbf{S}^1$, the characters of $\mathbf{S}^1$ are just the characters of $\mathbb{R}$ which are trivial on $\mathbb{Z}$. The result follows.

(3) If $\phi \in \widehat{\mathbb{Z}}$, then $\alpha = \phi(1) \in \mathbf{S}^1$ and $\phi(n) = \phi(1)^n = \alpha^n$.

(4) The characters of Z_k are the characters of $\mathbb{Z}$ that are trivial on $k\mathbb{Z}$, hence are of the form $\phi(n) = \alpha^n$ where α is a kth root of 1.

(5) Easy proof via the natural identification.

□

5.3. **The Fourier Transform.** We finally come to the definition of the Fourier transform. It will be convenient to employ a slightly different identification of $\widehat{G}$ with the maximal ideal space of $L^1(G)$ than the one given in (5.1). Namely, we shall associate to each $\xi \in \widehat{G}$, the functional

$$\xi \longrightarrow \overline{\xi}(f) = \xi^{-1}(f) = \int \overline{\langle x, \xi \rangle} f(x) dx$$

The Gelfand transform (here denoted as $\mathcal{F}$) on $L^1(G)$ then becomes the map from $L^1(G)$ to $C(\widehat{G})$ defined by

$$\mathcal{F} f(\xi) = \hat{f}(\xi) = \int \overline{\langle x, \xi \rangle} f(x) dx$$

Now using (4.1) as motivation, we define this map $\mathcal{F}$ to be the **Fourier transform** on G. Thus we have found analogues of the functions e^{int} in the case of $\mathbb{Z}$ and its dual group $\mathbf{S}^1$, for the general case of a locally compact Abelian G and its dual $\widehat{G}$. Those functions are nothing but the characters of G, that is, the elements of $\widehat{G}$.

The Fourier transform satisfies the following properties.

Theorem 5.9. *Let G be a locally compact Abelian group.*

(1) *The Fourier Transform is a norm-decreasing *-homomorphism from $L^1(G)$ to $C_0(G)$. It's range is a dense subspace of $C_0(\widehat{G})$.*
(2) *The Fourier transform of a translation is given by*

$$\widehat{L_y f}(\xi) = \int \overline{\langle x, \xi \rangle} f(y^{-1}x) dx = \overline{\langle y, \xi \rangle} \widehat{f}(\xi)$$

(3) *We also have*

$$\widehat{\eta f}(\xi) = \int \overline{\langle x, \xi \rangle} \langle x, \eta \rangle f(x) dx = \hat{f}(\eta^{-1}x) = L_\eta f(\xi)$$

Notice that (1) is the abstract formulation of the Riemann-Lebesgue lemma of classical Fourier Analysis.

Definition 5.10. A function $\phi \in L^\infty(G)$ on a locally compact group G is of **positive type** if it defines a positive linear functional on the Banach *-algebra $L^1(G)$, i.e., that satistfies

$$\int (f^* \star f)\phi \geq 0 \text{ for all } f \in L^1(G)$$

The set of all functions of positive type on G is denoted by $\mathcal{P}(G)$.

The Fourier transform can also be extended to complex Radon measures on G. But of more interest to us is a similar construction for measures on $\widehat{G}$. If $\mu \in M(\widehat{G})$, (where $M(\widehat{G})$ is the "measure algebra" on $\widehat{G}$, with product given by convolution) we define the continuous bounded function ϕ_μ on G by

$$\phi_\mu(x) = \int \langle x, \xi \rangle d\mu(\xi)$$

Theorem 5.11 (Bochner's Theorem)**.** *If $\phi \in \mathcal{P}(G)$, then there is a unique positive measure $\mu \in M(\widehat{G})$ such that $\phi = \phi_\mu = \int \langle x, \xi \rangle d\mu(\xi)$*

Bochner's theorem is useful in order to prove the Fourier Inversion Theorem. For a locally compact Abelian group G, let $\mathcal{B}(G) = \{\phi_\mu : \mu \in M(\widehat{G})\}$.

Theorem 5.12 (Fourier Inversion Formula I)**.** *If $f \in B(G) \cap L^1(G)$, then $\hat{f} \in L^1(\widehat{G})$, and if Haar measure $d\xi$ on $\widehat{G}$ is suitably normalized relative to the given Haar measure dx on G, we have $d\mu_f(\xi) = \hat{f}(\xi)d\xi$; that is;*

$$f(x) = \int \langle x, \xi \rangle \hat{f}(\xi) d\xi$$

When a Haar measure dx on G is given, a Haar measure $d\xi$ on $\widehat{G}$ that makes the above theorem true is called the **dual measure** of dx.

We have the following general result.

Theorem 5.13. *If G is compact and Haar measure is chosen so that $|G| = 1$, then the dual measure on $\widehat{G}$ is the counting measure. If G is discrete and Haar measure is chosen to be the counting measure, the dual measure on $\widehat{G}$ is one so that $|\widehat{G}| = 1$.*

Example 5.14. The groups $\mathbf{S}^1$ and $\mathbb{Z}$ are dual to each other, and if on $\mathbf{S}^1$ we choose the measure $1/2\pi d\theta$ to be the normalized Haar measure, then the counting measure on $\mathbb{Z}$ is the dual measure. The Fourier inversion theorem on these spaces now reads.

$$\hat{f}(n) = \int_0^{2\pi} f(\theta) e^{-in\theta} \frac{d\theta}{2\pi}$$

and

$$f(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{in\theta}$$

Example 5.15. If $G = Z_k$, the finite cyclic group has counting measure, its dual measure is the counting measure divided by k, and the Fourier inversion theorem reads:

$$\hat{f}(m) = \sum_0^k f(n) e^{-2\pi imn/k}$$

and

$$f(n) = \frac{1}{k} \sum_0^k \hat{f}(m) e^{2\pi imn/k}$$

We have now the fundamental theorem in the L^2 theory of the Fourier transform.

Theorem 5.16 (The Plancherel Theorem)**.** *For G Abelian, the Fourier transform on $L^1(G) \cup L^2(G)$ extends uniquely to a unitary isomorphism from $L^2(G)$ to $L^2(\widehat{G})$.*

Remark 5.17. If G was a compact Abelian group and $|G| = 1$, then $\widehat{G}$ is an orthonormal basis for $L^2(G)$.

Proof. We have that $\widehat{G}$ is an orthonormal set by Theorem 5.7 (5). Now if we take $f \in L^2(G)$ such that $0 = \int f\overline{\xi} = \hat{f}(\xi)$ for all ξ, then $f = 0$ by the Plancherel theorem. □

Theorem 5.18 (Hausdorff-Young Inequality)**.** *Suppose $1 < p \leq 2$ and $p^{-1} + q^{-1} = 1$ then we have that if $f \in L^p(G)$, then $\hat{f} \in L^q(\widehat{G})$ and $\|\hat{f}\|_q \leq \|f\|_p$*

5.4. **Pontryagin Duality.** For G a locally compact Abelian group, the elements of $\widehat{G}$ are characters on G, but we can equally well regard elements of G as characters on $\widehat{G}$. More precisely, each $x \in G$ defines a character $\Phi(x)$ on $\widehat{G}$ by

$$\langle \xi, \Phi(x) \rangle = \langle x, \xi \rangle$$

Φ is clearly a group homomorphism from G to $\widehat{\widehat{G}}$. It is a fundamental fact that Φ is actually an isomorphism. This is the Pontryagin Duality theorem.

Theorem 5.19 (Pontryagin Duality)**.** *The map $\Phi : G \to \widehat{\widehat{G}}$ defined above is an isomorphism of topological groups.*

As a corollary of the above theorem, we get another version of Fourier Inversion.

Theorem 5.20 (Fourier Inversion Theorem II)**.** *If $f \in L^1(G)$ and $\hat{f} \in L^1(\widehat{G})$, then $f(x) = \widehat{(\hat{f})}(x^{-1})$ for a.e x. That is,*

$$f(x) = \int \langle x, \xi \rangle \hat{f}(\xi) d\xi$$

for almost every x. If f is continuous, these relations hold for every x.

As a final conclusion, we state a structure theorem for locally compact Abelian groups.

Theorem 5.21 (Structure Theorem for Abelian Groups)**.** *Every locally compact Abelian group has an open subgroup of the form $\mathbb{R}^n \times G$ where G is a compact group.*

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