Constructing the Voevodsky universe in the Simplicial Model of HoTT.

Koundinya Vajjha

December 1, 2018

These notes are preparatory notes for a talk at the CMU HoTT seminar. I freely lift multiple passages from [1].

1 Type Theory.

The type theory we shall be starting out with is a (slight variant of) Martin-Lof's Intensional Type Theory - a dependent type theory, taking as basic constructors $\mathsf{\Pi}$-, $\mathsf{\Sigma}$-, Id-, and W-types, $\mathsf{0}$, $\mathsf{1}$, $+$, and one universe a la Tarski closed under these constructors.

A universe a la Tarski is a type together with an "interpretation" operation allowing us to regard its terms as types. This allows us to disambiguate between the *names of the types* and the types themselves, which is not the case in a Russell style universe.

1.1 How is such a type theory constructed?

It is done in two stages: first the *raw* or *untyped* syntax of the theory—the set of expressions that are at least parseable, but not necessarily meaningful—and then the *derivable judgements*, certain inductively-generated predicates picking out the genuinely meaningful contexts, types, and terms.

- The *raw syntax* may be constructed as certain strings of symbols, or alternatively, certain labelled trees. On this, one then defines α*-equivalence* (i.e., syntactic identity modulo renaming of bound variables), and the operation of *(capture-free) substitution*. (i.e., disallowing renaming of free variables in a λ by a variable which is bound in the λ-abstraction.)
- For the *derivable judgements* of the theory, one defines on the raw syntax several multi-place relations. For instance, "$\Gamma \vdash a : A$" will be a relation on triples (Γ, a, A) of a raw context, term, and type expression respectively, to be read as "a is a term of type A, in context Γ". These relations are defined by mutual induction, as the smallest family of relations closed under a bevy of specified closure conditions, the *inference rules* of the theory.

Both sorts of rules for the type theory in question are given in the appendix of [1].

2 Models of Type Theory – Contextual categories.

We have seen the different tools for ‘modelling‘ type theory, namely Categories with Attributes (CwA’s), Categories with Families (CwF’s) and in this talk, I will first introduce yet another such tool - Contextual Categories. Essentially, contextual categories are intended to provide a completely equivalent alternative to the syntactic presentation of type theory.

When I was at the IAS for a conference, Dan Grayson asked a gathering why we need syntax at all. Why couldn’t we just work on the semantic side? Conversely, why couldn’t one just work with the syntax and avoid the categorical semantics altogether?

An answer to the first question is that working with higher-order logical structure in contextual categories quickly becomes unreadable. [1] give the example of an unreadable version of function extensionality in Contextual Categories. And so one needs the syntax also, as a *notation* for what “actually goes on” in a contextual category.

For the second question, the trouble with syntax is that it is tricky to handle rigorously, since we must account for capture-free substitution, variable binding, multiple derivations of a judgment etc.

To motivate contextual categories, we look at the prototypical example of a type theory first - then abstract out the essential features. The first column describes the contextual category associated to a type theory **T**, which we denote as $\mathcal{C}(\mathbf{T})$. The second column describes the properties of an arbitrary contextual category.

Type Theory **T**/Contextual category $\mathcal{C}(\mathbf{T})$	A general contextual category $\mathcal{C}$
$\mathrm{Ob}_n\,\mathcal{C}(\mathbf{T})$ consists of the contexts $[x_1{:}A_1, \ldots, x_n{:}A_n]$ of length n, up to definitional equality and renaming of free variables	a grading of objects as $\mathrm{Ob}\,\mathcal{C} = \coprod_{n:\mathbb{N}} \mathrm{Ob}_n\,\mathcal{C}$
maps of $\mathcal{C}(\mathbf{T})$ are *context morphisms*, or *substitutions*, considered up to definitional equality and renaming of free variables.	
composition is given by substitution, and the identity $\Gamma \longrightarrow \Gamma$ by the variables of Γ, considered as terms;	
1 is the empty context [ ];	a unique object $1 \in \mathrm{Ob}_0\,\mathcal{C}$ which is the terminal object of $\mathcal{C}$
$\mathrm{ft}[x_1{:}A_1, \ldots, x_{n+1}{:}A_{n+1}] = [x_1{:}A_1, \ldots, x_n{:}A_n]$	maps $$\mathrm{ft}_n : \mathrm{Ob}_{n+1}\,\mathcal{C} \longrightarrow \mathrm{Ob}_n\,\mathcal{C}$$ for each n.

for $\Gamma = [x_1{:}A_1,\ \ldots,\ x_{n+1}{:}A_{n+1}]$, the map $p_\Gamma \colon \Gamma \longrightarrow \mathrm{ft}\,\Gamma$ is the *dependent projection* context morphism simply forgetting the last variable of Γ;	for each $X \in \mathrm{Ob}_{n+1}\,\mathcal{C}$, a map $p_X \colon X \longrightarrow \mathrm{ft}\,X$ (the *canonical projection* from X);
typed terms $\Gamma \vdash t : A$ of $\mathbf{T}$	sections of $p_{[\Gamma,\ x:A]} \colon [\Gamma,\ x{:}A] \longrightarrow \Gamma$ in $\mathcal{C}$
for contexts $$\Gamma = [x_1{:}A_1,\ \ldots,\ x_{n+1}{:}A_{n+1}(x_1, \ldots, x_n)],$$ $$\Gamma' = [y_1{:}B_1,\ \ldots,\ y_m{:}B_m(y_1, \ldots, y_{m-1})],$$ and a map $f = [f_i(\vec{y})]_{i \leq n} \colon \Gamma' \longrightarrow \mathrm{ft}\,\Gamma$, the pullback $f^*\Gamma$ is the context $$\begin{aligned}[&y_1{:}B_1,\ \ldots,\ y_m{:}B_m(y_1, \ldots, y_{m-1}),\\ &y_{m+1}{:}A_{n+1}(f_1(\vec{y}), \ldots, f_n(\vec{y}))],\end{aligned}$$ (for some fresh y_{m+1}) and $q(\Gamma, f) \colon f^*\Gamma \longrightarrow \Gamma$ is the map $$[f_1,\ \ldots, f_n,\ y_{m+1}].$$	for each $n > 0$, $X \in \mathrm{Ob}_{n+1}\,\mathcal{C}$ and $f \colon Y \longrightarrow \mathrm{ft}(X)$, an object $f^*(X)$ together with a map $q(f, X) \colon f^*(X) \longrightarrow X$ such that $\mathrm{ft}(f^*X) = Y$, and the square $$\begin{tikzcd} f^*(X) \arrow[r, "q(f{,}X)"] \arrow[d, "p_{f^*X}"'] & X \arrow[d, "p_x"] \\ Y \arrow[r, "f"] & \mathrm{ft}(X) \end{tikzcd}$$ is a pullback.
syntactic substitution is strictly associative	the canonical pullbacks $q(f, X)$ are strictly functorial.

If our type theory $\mathbf{T}$ has logical structure, for example, a Π-structure, we can mimic that in contextual category $\mathcal{C}$ as $\mathsf{\Pi}$-structure:

Type theory $\mathbf{T}$	Contextual Category $\mathcal{C}$
$$\dfrac{\Gamma,\ x{:}A \vdash B(x)\ \mathsf{type}}{\Gamma \vdash \mathsf{\Pi}_{x:A} B(x)\ \mathsf{type}}\ \mathsf{\Pi}\text{-}\textsc{form}$$	for each $(\Gamma, A, B) \in \mathrm{Ob}_{n+2}\,\mathcal{C}$,there is an object $(\Gamma, \mathsf{\Pi}(A, B)) \in \mathrm{Ob}_{n+1}\,\mathcal{C}$;
$$\dfrac{\begin{array}{c}\Gamma,\ x{:}A \vdash B(x)\ \mathsf{type}\\ \Gamma,\ x{:}A \vdash b(x) : B(x)\end{array}}{\Gamma \vdash \lambda x{:}A.b(x) : \mathsf{\Pi}_{x:A} B(x)}\ \mathsf{\Pi}\text{-}\textsc{intro}$$	for each such (Γ, A, B) and section $b \colon (\Gamma, A) \longrightarrow (\Gamma, A, B)$, a section $\lambda(b) \colon \Gamma \longrightarrow (\Gamma, \mathsf{\Pi}(A, B))$;

$\dfrac{\Gamma \vdash f{:}\mathsf{\Pi}_{x:A}B(x) \qquad \Gamma \vdash a : A}{\Gamma \vdash \mathsf{app}(f,a) : B(a)}\ \mathsf{\Pi}\text{-}\textsc{app}$	for each such (Γ, A, B) and pair of sections $k\colon \Gamma \longrightarrow (\Gamma, \mathsf{\Pi}(A,B))$, $a\colon \Gamma \longrightarrow (\Gamma, A)$, a section $\mathsf{app}(k,a)\colon \Gamma \longrightarrow (\Gamma, A, B)$ such that $p_B \cdot \mathsf{app}(k,a) = a$. $$\begin{tikzcd} & (\Gamma, A, B) \arrow[d, "p_B"] \\ & (\Gamma, A) \arrow[d, "p_A"] \\ \Gamma \arrow[uur, bend left, "\mathsf{app}(k,a)"] \arrow[ur, "a"] \arrow[r, equal] & \Gamma; \end{tikzcd}$$
$\dfrac{\begin{array}{c}\Gamma,\ x{:}A \vdash B(x)\ \mathsf{type} \\ \Gamma,\ x{:}A \vdash b(x) : B(x) \\ \Gamma \vdash a : A\end{array}}{\Gamma \vdash \mathsf{app}(\lambda x{:}A.b(x), a) = b(a) : B(a)}\ \mathsf{\Pi}\text{-}\textsc{comp}$	such that for each (Γ, A, B), $a\colon \Gamma \longrightarrow (\Gamma, A)$ and $b\colon (\Gamma, A) \longrightarrow (\Gamma, A, B)$, we have $\mathsf{app}(\lambda(b), a) = b \cdot a$;

And similarly for $\mathsf{\Sigma}$,W,Id,$\mathsf{1}$,$\mathsf{0}$,$+$-structures on $\mathcal{C}$.

Remark 2.0.1. For completeness, we define things left implicit in the above table:

1. In a type theory $\mathbf{T}$ a *context morphism*

   $$f\colon [x_1{:}A_1,\ \ldots,\ x_n{:}A_n] \longrightarrow [y_1{:}B_1,\ \ldots,\ y_m{:}B_m(y_1, \ldots, y_{m-1})]$$

   is an equivalence class of sequences of terms $f_1, \ldots, f_m$ such that

   $$\begin{gathered} x_1{:}A_1,\ \ldots,\ x_n{:}A_n \vdash f_1 : B_1 \\ \vdots \\ x_1{:}A_1,\ \ldots,\ x_n{:}A_n \vdash f_m : B_m(f_1,\ \ldots,\ f_{m-1}), \end{gathered}$$

   and two such maps $[f_i]$, $[g_i]$ are equal exactly if for each i,

   $$x_1{:}A_1,\ \ldots,\ x_n{:}A_n \vdash f_i = g_i \colon B_i(f_1,\ \ldots\ f_{i-1});$$

   That is, for each type in the codomain context, f constructs a term of that type out of that data in the domain.

2. Strict functoriality of $q(f, X)$ means that for $X \in \mathrm{Ob}_{n+1}\mathcal{C}$, $1^*_{\mathrm{ft}\,X}X = X$ and $q(1_{\mathrm{ft}\,X}, X) = 1_X$; and for $X \in \mathrm{Ob}_{n+1}\mathcal{C}$, $f\colon Y \longrightarrow \mathrm{ft}\,X$ and $g\colon Z \longrightarrow Y$, we have $(fg)^*(X) = g^*(f^*(X))$ and $q(fg, X) = q(f, X)q(g, f^*X)$.

One can also define the notion of a contextual functor in the obvious way. The idea is therefore that given any contextual category $\mathscr{C}$ with structure corresponding to the logical rules of some syntactic type theory $\mathbb{T}$, one should obtain an interpretation of the syntax of $\mathbb{T}$ in $\mathscr{C}$; and in proving this, one deals with the subtleties and bureaucracy of $\mathbb{T}$ once and for all, giving a clear framework for subsequently constructing models of $\mathbb{T}$. This is made precise in the "initiality conjecture" of Voevodsky: the contextual category $\mathcal{C}(\mathbf{T})$ as defined in the first table is *initial* in the category of all contextual categories for a type theory $\mathbf{T}$.

But such an "initiality theorem" has only been proven in small type theories, and has not yet been proven for the type theory we set out with. The general prevailing view seemed to be that the initiality theorem for the type theory we set out with is a straightforward extension of existing initiality proofs and so the matter was more or less taken for granted as true. Voevodsky argued that this was not rigourous and unsatisfactory and so now there is a community effort, led by Mike Shulman, to prove the initiality conjecture to settle it once and for all.

3 Contextual Categories from Universes

A reason for looking at Contextual Categories is because they can be generated by "universes" (this was realized by Voevodsky) and so to generate the data of a Contextual Category, one only needs to consider the (much simpler) definition of a universe. Universes also provide a novel way to resolve the "coherence problem": the requirement for the pullbacks to be strictly functorial.

Definition 3.0.1. Let $\mathcal{C}$ be a category. A *universe* in $\mathcal{C}$ is an object U together with a morphism $p\colon \tilde{U} \longrightarrow U$, and for each map $f\colon X \longrightarrow U$ a choice of pullback square (meaning that we pick *one* representative from the equivalence class of isomorphic pullbacks, and stick with it)

$$\begin{tikzcd}
(X;f) \arrow[r, "Q(f)"] \arrow[d, "P_{(X,f)}"'] & \tilde{U} \arrow[d, "p"] \\
X \arrow[r, "f"] & U
\end{tikzcd}$$

The intuition behind the definition of the universe is that $\tilde{U}$ stands for the collection of all *abstract terms* of a type theory, and U stands for the collection of all *names of the types* in the type theory. The map p then associates every term to a type, which is required of any term in type theory. We refer to a universe simply as U, with the chosen pullbacks and p understood.

Given a map $q\colon Y \longrightarrow X$, we will often write $\ulcorner q \urcorner$ (or $\ulcorner Y \urcorner$, if q is understood) for a map $X \longrightarrow U$ such that $q \cong P_{(X,\ulcorner q \urcorner)}$ in $\mathcal{C}/X$. Also, for a sequence of maps $f_1\colon X \longrightarrow U$, $f_2\colon (X;f_1) \longrightarrow U$, etc., we write $(X; f_1, \ldots, f_n)$ for $((\ldots(X;f_1);\ldots);f_n)$. (In particular, with $n = 0$, $(X;\ ) = X$.)

Definition 3.0.2. Given a category $\mathcal{C}$, together with a universe U and a terminal object 1, we define a contextual category $\mathcal{C}_U$ as follows:

- $\mathrm{Ob}_n\,\mathcal{C}_U := \{\ (f_1, \dots, f_n) \in (\mathrm{Mor}\mathcal{C})^n \mid f_i \colon (1; f_1, \dots, f_{i-1}) \longrightarrow U\ (1 \leq i \leq n)\ \}$;
- $\mathcal{C}_U((f_1, \dots, f_n), (g_1, \dots, g_m)) := \mathcal{C}((1; f_1, \dots, f_n), (1; g_1, \dots, g_n))$;
- $1_{\mathcal{C}_U} := ()$, the empty sequence;
- $\mathrm{ft}(f_1, \dots, f_{n+1}) := (f_1, \dots, f_n)$;
- the projection $p_{(f_1,\dots,f_{n+1})}$ is the map $P_{(X,f_{n+1})}$ provided by the universe structure on U;
- given $(f_1, \dots, f_{n+1})$ and a map $\alpha \colon (g_1, \dots, g_m) \longrightarrow (f_1, \dots, f_n)$ in $\mathcal{C}_U$, the canonical pullback $\alpha^*(f_1, \dots, f_{n+1})$ in $\mathcal{C}_U$ is given by $(g_1, \dots, g_m,\, f_{n+1} \cdot \alpha)$, with projection induced by $Q(f_{n+1} \cdot \alpha)$:

$$\begin{tikzcd}
(1; g_1, \dots, g_m, f_{n+1} \cdot \alpha) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rr, bend left, "Q(f_{n+1} \cdot \alpha)"] & (1; f_1, \dots, f_{n+1}) \arrow[r, "Q(f_{n+1})"'] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U} \arrow[d, "p"] \\
(1; g_1, \dots, g_m) \arrow[r, "\alpha"] & (1; f_1, \dots, f_n) \arrow[r, "f_{n+1}"] & U
\end{tikzcd}$$

Building on the intuition that if U stands for all "names of types", then $f : 1 \to U$ represents a single name of a type, named $f(1)$. Then the pullback $(1; f)$

$$\begin{tikzcd}
(1; f) \arrow[r, "Q(f)"] \arrow[d, "P_{(1,f)}"'] & \tilde{U} \arrow[d, "p"] \\
1 \arrow[r, "f"] & U
\end{tikzcd}$$

represents all terms whose type is $f(1)$. Now, a map $g : (1; f) \to U$ takes a term of type $f(1)$ and spits out a type. In other words, the map g represents a dependent family of types over $f(1)$.

$$\begin{tikzcd}
(1; f, g) \arrow[r, "Q(g)"] \arrow[d, "P_{(1;f),g}"'] & \tilde{U} \arrow[d, "p"] \\
(1; f) \arrow[r, "g"] & U
\end{tikzcd}$$

Arguing further, the pullback $(1; f, g)$ represents all terms of a dependent family of types over $f(1)$. This then provides a basis for what $(1; f_1, \dots, f_n)$ stand for, intuitively.

Proposition 3.0.3.

1. *These data define a contextual category* $\mathcal{C}_U$.
2. *This contextual category is well-defined up to canonical isomorphism given just* $\mathcal{C}$ *and* $p \colon \tilde{U} \longrightarrow U$*, independently of the choice of pullbacks and terminal object.*

Proof. Routine computation. ☐

Assuming this proposition, we can now focus on equipping *the universe* with logical structure. Here is a handy dictionary between judgements and their corresponding statements in a category with a universe:

Type theory $\mathbf{T}$	Category $\mathcal{C}$ with a universe
$\vdash A : \mathsf{type}$	$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U} \arrow[d, "p"] \\ 1 \arrow[r, "\ulcorner A \urcorner"] & U \end{tikzcd}$
$\Gamma \vdash A : \mathsf{type}$	$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U} \arrow[d, "p"] \\ \Gamma \arrow[r, "\ulcorner A \urcorner"] & U \end{tikzcd}$
$\Gamma \vdash a : A$	$\begin{tikzcd} A \arrow[r] \arrow[d, "P_{\Gamma,\ulcorner A \urcorner}"] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U} \arrow[d, "p"] \\ \Gamma \arrow[r, "\ulcorner A \urcorner"] \arrow[u, bend left, "s"] & U \end{tikzcd}$ s a section
$\Gamma, x : A \vdash B : \mathsf{type}$	$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U} \arrow[d, "p"] \\ \Gamma \arrow[r, "\ulcorner A \urcorner"] & U \end{tikzcd}$ $\begin{tikzcd} B \arrow[r] \arrow[d, "P_{A,\ulcorner B \urcorner}"] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U} \arrow[d, "p"] \\ A \arrow[r, "\ulcorner B \urcorner"] & U \end{tikzcd}$
$\Gamma, x : A \vdash b(x) : B(x)$	$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U} \arrow[d, "p"] \\ \Gamma \arrow[r, "\ulcorner A \urcorner"] & U \end{tikzcd}$ $\begin{tikzcd} B \arrow[r] \arrow[d, "P_{A,\ulcorner B \urcorner}"] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U} \arrow[d, "p"] \\ A \arrow[r, "\ulcorner B \urcorner"] \arrow[u, bend left, "s"] & U \end{tikzcd}$

4 Logical structure on Universes

Given a universe U in a category $\mathcal{C}$, we want to know how to equip the contextual category $\mathcal{C}_U$ with various logical structure—$\mathsf{\Pi}$-types, $\mathsf{\Sigma}$-types,W-types,Id-types,$\mathsf{1}$-type,$\mathsf{0}$-type,$+$-type and the universe type. Assuming that $\mathcal{C}$ is locally cartesian closed, in [1] there is given the required structure on U to generate the corresponding logical structure on $\mathcal{C}_U$.

As an example, we shall equip our category U with the $+$-structure in order to generate a $+$-structure on $\mathcal{C}_U$.

Definition 4.0.1. A $+$-*structure* on U consists of a map $+\colon U \times U \longrightarrow U$, together with an isomorphism $+^*\tilde{U} \cong \pi_1^*\tilde{U} + \pi_2^*\tilde{U}$ in $\mathcal{C}/(U \times U)$.

Theorem 4.0.2. *A $+$-structure on a universe U induces $+$-type structure on $\mathcal{C}_U$.*

Proof. We start out with a category $\mathcal{C}$ equipped with a universe structure and a $+$-structure. Given this, we want to equip the contextual category with a $+$-structure. Let us recall the correspondence between the $+$-type constructor in type theory and $+$-structure in a contextual category:

Type theory $\mathbf{T}$	Contextual Category $\mathcal{C}$
$\dfrac{\Gamma \vdash A\ \mathsf{type} \qquad \Gamma \vdash B\ \mathsf{type}}{\Gamma \vdash A + B\ \mathsf{type}}\ + - \textsc{form}$	for any objects (Γ, A) and (Γ, B), an object $(\Gamma, A + B)$;
$\dfrac{\Gamma \vdash A\ \mathsf{type} \qquad \Gamma \vdash B\ \mathsf{type}}{\Gamma,\ x{:}A \vdash \mathsf{inl}(x) : A + B}\ +\text{-}\textsc{intro}\ 1.$	for each such (Γ, A), (Γ, B), maps $\mathsf{inl}_{A,B}\colon (\Gamma, A) \longrightarrow (\Gamma, A + B)$, over Γ;
$\dfrac{\Gamma \vdash A\ \mathsf{type} \qquad \Gamma \vdash B\ \mathsf{type}}{\Gamma,\ y{:}B \vdash \mathsf{inr}(y) : A + B}\ +\text{-}\textsc{intro}\ 2.$	for each such (Γ, A), (Γ, B), a map $\mathsf{inr}_{A,B}\colon (\Gamma, B) \longrightarrow (\Gamma, A + B)$, over Γ;
$\dfrac{\begin{array}{c}\Gamma,\ z{:}A + B \vdash C(z)\ \mathsf{type} \\ \Gamma,\ x{:}A \vdash d_l(x) : C(\mathsf{inl}(x)) \\ \Gamma,\ y{:}B \vdash d_r(y) : C(\mathsf{inr}(y))\end{array}}{\Gamma,\ z{:}A + B \vdash \mathsf{case}_{d_l,d_r}(z) : C(z)}\ +\text{-}\textsc{elim}$	for each object $(\Gamma, A + B, C)$, and maps $d_l\colon (\Gamma, A) \longrightarrow (\Gamma, A + B, C)$, $d_r\colon (\Gamma, B) \longrightarrow (\Gamma, A + B, C)$ with $p_C \cdot d_l = \mathsf{inl}_{A,B}$ and $p_C \cdot d_r = \mathsf{inr}_{A,B}$, a section $\mathsf{case}_{C,d_l,d_r}\colon (\Gamma, A + B) \longrightarrow (\Gamma, A + B, C)$;
$\dfrac{\begin{array}{c}\Gamma,\ z{:}A + B \vdash C(z)\ \mathsf{type} \\ \Gamma,\ x{:}A \vdash d_l(x) : C(\mathsf{inl}(x)) \\ \Gamma,\ y{:}B \vdash d_r(y) : C(\mathsf{inr}(y))\end{array}}{\Gamma,\ x{:}A \vdash \mathsf{case}_{d_l,d_r}(\mathsf{inl}(x)) = d_l(x) : C(\mathsf{inl}(x))}$	such that $\mathsf{case}_{C,d_l,d_r} \cdot \mathsf{inl}_{A,B} = d_l$,

$\dfrac{\begin{array}{c}\Gamma,\ z{:}A+B \vdash C(z)\ \mathsf{type} \\ \Gamma,\ x{:}A \vdash d_l(x) : C(\mathsf{inl}(x)) \\ \Gamma,\ y{:}B \vdash d_r(y) : C(\mathsf{inr}(y))\end{array}}{\Gamma,\ y{:}B \vdash \mathsf{case}_{d_l,d_r}(\mathsf{inr}(y)) = d_r(y) : C(\mathsf{inr}(y))}$	and $\mathsf{case}_{C,d_l,d_r} \cdot \mathsf{inr}_{A,B} = d_r$

The proof is essentially a routine verification:

+-FORM: The premises

$$\Gamma \vdash A\ \mathsf{type} \qquad \Gamma \vdash B\ \mathsf{type}$$

correspond to diagrams:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U} \arrow[d] & & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U} \arrow[d] \\
\Gamma \arrow[r, "\ulcorner A \urcorner"] & U & & \Gamma \arrow[r, "\ulcorner B \urcorner"] & U
\end{tikzcd}$$

and hence we have a composite map $\Gamma \xrightarrow{(\ulcorner A \urcorner, \ulcorner B \urcorner)} U \times U \xrightarrow{+} U$, which we take as $\ulcorner A + B \urcorner$. This is justified by the following computation:

$$\begin{aligned}
(+ \circ (\ulcorner A \urcorner, \ulcorner B \urcorner))^* \tilde{U} &\cong ((\ulcorner A \urcorner, \ulcorner B \urcorner)^* \circ +^*)\tilde{U} \\
&\cong (\ulcorner A \urcorner, \ulcorner B \urcorner)^* (\pi_1^* \tilde{U} + \pi_2^* \tilde{U}) \\
&\cong (\ulcorner A \urcorner, \ulcorner B \urcorner)^* (\pi_1^* \tilde{U}) + (\ulcorner A \urcorner, \ulcorner B \urcorner)^* (\pi_2^* \tilde{U}) \\
&\cong (\pi_1 \circ (\ulcorner A \urcorner, \ulcorner B \urcorner))^* \tilde{U} + (\pi_2 \circ (\ulcorner A \urcorner, \ulcorner B \urcorner))^* \tilde{U} \\
&\cong (\ulcorner A \urcorner)^* \tilde{U} + (\ulcorner B \urcorner)^* \tilde{U} \\
&\cong A + B
\end{aligned}$$

So, $A + B$ is attained as pullback of $p : \tilde{U} \to U$ along $+ \circ (\ulcorner A \urcorner, \ulcorner B \urcorner)$ and hence it is an object of the contextual category $\mathcal{C}_U$.

+-INTRO: As we've seen above, given objects A, B in $\mathcal{C}$, the +-structure on the universe gives us an object $A + B$ in $\mathcal{C}$ and so the maps $\mathsf{inl}_{A,B} : A \to A + B$ and $\mathsf{inr}_{A,B} : B \to A + B$ are simply the the canonical injections $A \to A + B$ and $B \to A + B$.

+-ELIM, +-COMP: We are given the following data:

$$\begin{tikzcd}
C \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & \tilde{U} \arrow[d] \\
A + B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rr, bend left] & \tilde{U} \arrow[d] & U \\
\Gamma \arrow[r] & U &
\end{tikzcd}$$

We are also given $p_C \circ d_r = \mathsf{inr}$ and $p_C \circ d_l = \mathsf{inl}$ in the following picture:

$$\begin{tikzcd}
& C \arrow[d, bend right, "p_C"'] & \\
A \arrow[ur, bend left, "d_l"] \arrow[r, "\mathsf{inl}"'] & A + B \arrow[u, dashed, "\exists!\mathsf{case}"'] & B \arrow[ul, bend right, "d_r"'] \arrow[l, "\mathsf{inr}"]
\end{tikzcd}$$

So by the universal property of the coproduct, there is a unique map $\mathsf{case} : A + B \to C$ such that $\mathsf{case} \circ \mathsf{inl} = d_l$ and $\mathsf{case} \circ \mathsf{inr} = d_r$, which is exactly what we wanted. Also, we can clearly see that case is a section of p_C.

$\square$

There are similar theorems for other logical structures, and to derive them all, we need to assume that the category on which we define the universe is locally cartesian closed.

5 A universe of Kan Complexes

The simplicial category $\mathbf{\Delta}$ is the category whose objects are natural numbers (denoted $[n]$) and morphisms from $[m]$ to $[n]$ are order preserving maps from the finite set $\{0, \ldots, m\}$ to the finite set $\{0, \ldots, n\}$. This category is generated by the coface and codegeneracy maps $d^i : [n-1] \to [n]$ and $s^j : [n+1] \to [n]$.

Definition 5.0.1. A *simplicial set* is a contravariant functor $\mathbf{\Delta}^{op} \to \mathcal{S}\mathrm{et}$. The category of simplicial sets is denoted $\mathrm{s}\mathcal{S}\mathrm{et}$. It is a Grothendieck topos and in particular an lccc. The n-simplices of a simplicial set X are denoted X_n. So write down the data for a simplicial set X, it is enough to write down the sets X_n and the maps $d_i : X_n \to X_{n-1}$ and $s_j : X_{n+1} \to X_n$ (satisfying the simplicial identities).

The standard n-simplex is defined to be the simplicial set

$$\Delta^n = \mathrm{Hom}_{\mathbf{\Delta}}(, [n])$$

The "top cell" or the "standard simplex" is the identity element $\iota_n \in \Delta^n_n = \mathrm{Hom}_{\mathbf{\Delta}}([n], [n])$.

Definition 5.0.2. The k-th horn of the standard n-simplex is the subsimplicial set $\Lambda^n_k \subset \Delta^n$ which is generated by the faces $d_j(\iota_n)$ for $j \neq k$. (It contains all of the listed simplices, plus all of their iterated degeneracies).

Definition 5.0.3. There is a *weak factorization system* on $\mathrm{s}\mathcal{S}\mathrm{et}$ consisting of $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ (this means that (all maps in) $\mathcal{C} \cap \mathcal{W}$ has the left lifting property with respect to $\mathcal{F}$, $\mathcal{F}$ has right lifting property with respect to $\mathcal{C} \cap \mathcal{W}$ and every map can be factored as a composite of a map in $\mathcal{F}$ followed by a map in $\mathcal{C} \cap \mathcal{W}$) where:

- $\mathcal{C}$ are the class of all *cofibrations*, consisting of monomorphisms in $\mathrm{s}\mathcal{S}\mathrm{et}$.

- $\mathcal{W}$ are the class of all *weak equivalences*, consisting of those maps $f : X \to Y$ such that $|f| : |X| \to |Y|$ is a weak homotopy equivalence.
- $\mathcal{F}$ are the class of all *fibrations*, called *Kan fibrations*, which are all maps $p : E \to B$ such that all such lifting problems can be solved.

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\
\Delta[n] \arrow[r] \arrow[ur] & B
\end{tikzcd}$$

The category s$\mathcal{S}$et provides a "model" for the ordinary homotopy theory of topological spaces. More precisely, the homotopy category of s$\mathcal{S}$et and the homotopy category of compactly generated weak hausdorff spaces **CGWH** are Quillen equivalent. Because of this fact one can, when working up to homotopy, think of simplicial sets as of combinatorial representations of shapes or spaces, of simplicial paths as paths on these spaces etc. This provides a lot of the underlying intuition for the simplicial models.

With this machinery set up, we come to the main topic of the paper - a model of our type theory in s$\mathcal{S}$et. Type dependency is modelled by Kan fibrations, and closed types as Kan complexes. We will need a bunch of preliminary definitions:

Definition 5.0.4. An infinite cardinal κ is called a *regular cardinal* if the category $\mathcal{S}\mathrm{et}_{<\kappa}$ of sets of cardinality smaller than κ has all colimits or equivalently, given a function $P \to X$ regarded as a family $\{P_x\}_{x\in X}$ of sets such that $|X| < \kappa$ and $|P_x| < \kappa$ for all $x \in X$, then $|P| < \kappa$.

A cardinal number κ is called an *inaccessible cardinal* if it cannot be "accessed" from smaller cardinals using basic operations of sum and powerset.

In a word, *regular* - closed under sums and *inaccessible* - closed under sums and powerset.

Our aim now is to construct, for any regular cardinal α, a Kan fibration $p_\alpha : \widetilde{\mathrm{U}}_\alpha \to \mathrm{U}_\alpha$, 'weakly universal' among Kan fibrations with α-small fibers.

Definition 5.0.5. A *well-ordered morphism* of simplicial sets consists of an ordinary map of simplicial sets $f \colon Y \longrightarrow X$, together with a function assigning to each simplex $x \in X_n$ a well-ordering on the fiber $Y_x := f^{-1}(x) \subseteq Y_n$.

If $f \colon Y \longrightarrow X$, $f' \colon Y' \longrightarrow X$ are well-ordered morphisms into a common base X, an *isomorphism* of well-ordered morphisms from f to f' is an isomorphism $Y \cong Y'$ over X preserving the well-orderings on the fibers.

Proposition 5.0.6. *Given two well-ordered sets, there is at most one isomorphism between them. Given two well-ordered morphisms over a common base, there is at most one isomorphism between them.*

Proof. The first statement is classical (and immediate by induction); the second follows from the first, applied in each fiber. ☐

Definition 5.0.7. Fix (for the remainder of this and the following section) a regular cardinal α. Say a map of simplicial sets $f \colon Y \longrightarrow X$ is *α-small* if each of its fibers Y_x has cardinality $< \alpha$.

Given a simplicial set X, define $\mathbf{W}_\alpha(X)$ to be the set of isomorphism classes of α-small well-ordered morphisms $Y \longrightarrow X$; together with the pullback action $\mathbf{W}_\alpha(f) := f^* \colon \mathbf{W}_\alpha(X) \longrightarrow \mathbf{W}_\alpha(X')$, for $f \colon X' \longrightarrow X$, this gives a contravariant functor $\mathbf{W}_\alpha \colon s\mathcal{S}\mathrm{et}^{\mathrm{op}} \longrightarrow \mathcal{S}\mathrm{et}$.

Lemma 5.0.8. *$\mathbf{W}_\alpha$ preserves all limits: $\mathbf{W}_\alpha(\mathrm{colim}_i X_i) \cong \lim_i \mathbf{W}_\alpha(X_i)$.*

Proof. Suppose $F \colon \mathcal{I} \longrightarrow s\mathcal{S}\mathrm{et}$ is some diagram, and $X = \mathrm{colim}_{\mathcal{I}} F$ is its colimit, with injections $\nu_i \colon F(i) \longrightarrow X$. We need to show that the canonical map $\mathbf{W}_\alpha(X) \longrightarrow \lim_{\mathcal{I}} \mathbf{W}_\alpha(F(i))$ is an isomorphism.

To see that it is surjective, suppose we are given $[f_i \colon Y_i \longrightarrow F(i)] \in \lim_{\mathcal{I}} \mathbf{W}_\alpha(F(i))$. For each $x \in X_n$, choose some i and $\bar{x} \in F(i)$ with $\nu(\bar{x}) = x$, and set $Y_x := (Y_i)_{\bar{x}}$. By Proposition 5.0.6, this is well-defined up to *canonical* isomorphism, independent of the choices of representatives i, $\bar{x}$, Y_i, f_i. The total space of these fibers then defines a well-ordered morphism $f \colon Y \longrightarrow X$, with fibers of size $< \alpha$, and with pullbacks isomorphic to f_i as required.

For injectivity, suppose f, f' are well-ordered morphisms over X, and $\nu_i^* f \cong \nu_i^* f'$ for each i. By Proposition 5.0.6, these isomorphisms must agree on each fiber, so together give an isomorphism $f \cong f'$. ☐

Define the simplicial set W_α by

$$\mathrm{W}_\alpha := \mathbf{W}_\alpha \cdot \mathbf{y}^{\mathrm{op}} \colon \Delta^{\mathrm{op}} \longrightarrow \mathcal{S}\mathrm{et},$$

where $\mathbf{y}$ denotes the Yoneda embedding $\Delta \longrightarrow s\mathcal{S}\mathrm{et}$.

Lemma 5.0.9. *The functor $\mathbf{W}_\alpha$ is representable, represented by W_α.*

Proof. The functors $\mathbf{W}_\alpha$ and $\mathrm{Hom}(-, \mathrm{W}_\alpha)$ agree up to isomorphism on the standard simplices (by the Yoneda lemma), and send colimits in $s\mathcal{S}\mathrm{et}$ to limits; but every simplicial set is canonically a colimit of standard simplices. ☐

Notation 5.0.10. Given an α-small well-ordered map $f \colon Y \longrightarrow X$, the corresponding map $X \longrightarrow \mathrm{W}_\alpha$ will be denoted by $\ulcorner f \urcorner$.

Applying the natural isomorphism above to the identity map $\mathrm{W}_\alpha \longrightarrow \mathrm{W}_\alpha$ yields a universal α-small well-ordered simplicial set $\widetilde{\mathrm{W}}_\alpha \longrightarrow \mathrm{W}_\alpha$. Explicitly, n-simplices of $\widetilde{\mathrm{W}}_\alpha$ are classes of pairs

$$(f \colon Y \longrightarrow \Delta[n], s \in f^{-1}(1_{[n]}))$$

i.e. the fiber of $\widetilde{\mathrm{W}}_\alpha$ over an n-simplex $\ulcorner f \urcorner \in \mathrm{W}_\alpha$ is exactly (an isomorphic copy of) the main fiber of f. So, by construction:

Proposition 5.0.11. *The canonical projection $\widetilde{\mathrm{W}}_\alpha \longrightarrow \mathrm{W}_\alpha$ is strictly universal for α-small well-ordered morphisms; that is, any such morphism can be expressed uniquely as a pullback of this projection.* ☐

Corollary 5.0.12. *The canonical projection* $\widetilde{\mathrm{W}}_\alpha \longrightarrow \mathrm{W}_\alpha$ *is weakly universal for* α*-small morphisms of simplicial sets: any such morphism can be given, not necessarily uniquely, as a pullback of this projection.*

Proof. By the well-ordering principle and the axiom of choice, one can well-order the fibers, and then use the universal property of W_α. ☐

Definition 5.0.13. Let $\mathbf{U}_\alpha \subseteq \mathbf{W}_\alpha$ (respectively, $\mathrm{U}_\alpha \subseteq \mathrm{W}_\alpha$) be the subobject consisting of (isomorphism classes of) α-small well-ordered fibrations[1]; and define $p_\alpha \colon \widetilde{\mathrm{U}}_\alpha \longrightarrow \mathrm{U}_\alpha$ as the pullback:

$$\begin{tikzcd}
\widetilde{\mathrm{U}}_\alpha \arrow[r] \arrow[d, "p_\alpha"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathrm{W}}_\alpha \arrow[d] \\
\mathrm{U}_\alpha \arrow[r, hook] & \mathrm{W}_\alpha
\end{tikzcd}$$

Lemma 5.0.14. *The map* $p_\alpha \colon \widetilde{\mathrm{U}}_\alpha \longrightarrow \mathrm{U}_\alpha$ *is a fibration.*

Proof. Consider a horn to be filled

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r] \arrow[d, hook] & \widetilde{\mathrm{U}}_\alpha \arrow[d, "p_\alpha"] \\
\Delta[n] \arrow[r, "\ulcorner x \urcorner"] & \mathrm{U}_\alpha
\end{tikzcd}$$

for some $0 \leq k \leq n$. It factors through the pullback

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r] \arrow[d, hook] & \bullet \arrow[r] \arrow[d, "x"] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathrm{U}}_\alpha \arrow[d, "p_\alpha"] \\
\Delta[n] \arrow[r, equal] & \Delta[n] \arrow[r, "\ulcorner x \urcorner"] & \mathrm{U}_\alpha
\end{tikzcd}$$

where by the definition of U_α and $\widetilde{\mathrm{U}}_\alpha$, x is a fibration. Thus the left square admits a diagonal filler, and hence so does the outer rectangle. ☐

Lemma 5.0.15. *An* α*-small well-ordered morphism* $f \colon Y \longrightarrow X \in \mathbf{W}_\alpha(X)$ *is a fibration if and only if* $\ulcorner f \urcorner \colon X \longrightarrow \mathrm{W}_\alpha$ *factors through* U_α.

Proof. For '⇒', assume that $f \colon Y \longrightarrow X$ is a fibration. Then the pullback of f to any representable is certainly a fibration:

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "x^* f"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\
\Delta[n] \arrow[r, "x"] & X.
\end{tikzcd}$$

---

[1] Here and throughout, by "fibration" we always mean "Kan fibration".

so $\ulcorner f \urcorner(x) = \ulcorner x^* f \urcorner \in \mathrm{U}_\alpha$, and hence $\ulcorner f \urcorner$ factors through U_α.

Conversely, suppose $\ulcorner f \urcorner$ factors through U_α. Then we obtain:

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathrm{U}}_\alpha \arrow[r] \arrow[d, "p_\alpha"] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathrm{W}}_\alpha \arrow[d] \\
X \arrow[r] & \mathrm{U}_\alpha \arrow[r, hook] & \mathrm{W}_\alpha,
\end{tikzcd}$$

where the lower composite is $\ulcorner f \urcorner$, and the outer rectangle and the right square are by construction pullbacks. Hence so is the left square; so by Lemma 5.0.14 f is a fibration. □

Corollary 5.0.16. *The functor* $\mathbf{U}_\alpha$ *is representable, represented by* U_α*; so* $p_\alpha \colon \widetilde{\mathrm{U}}_\alpha \longrightarrow \mathrm{U}_\alpha$ *is strictly universal for* α*-small well-ordered fibrations, and weakly universal for* α*-small fibrations.*

TO-DO:

- The simplicial set U_α is a **Kan complex**.
- The Kan fibration $p_\alpha : \widetilde{\mathrm{U}}_\alpha \longrightarrow \mathrm{U}_\alpha$ is **univalent**.

References

[1] Chris Kapulkin and Peter LeFanu Lumsdaine. The simplicial model of univalent foundations (after voevodsky). *arXiv preprint arXiv:1211.2851*, 2012.