Overview of Homotopy Theory

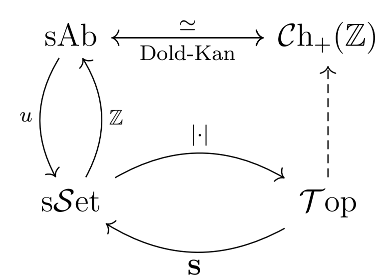
Koundinya Vajjha

December 2018

The following document is an outline of modern homotopy theory following a big-picture outline as given in one of Rick Jardine's courses on the subject. Any errors are my own.

The simplicial category $\mathbf{\Delta}$ is the category whose objects are natural numbers (denoted $[n]$) and morphisms from $[m]$ to $[n]$ are order preserving maps from the finite set $\{0, \ldots, m\}$ to the finite set $\{0, \ldots, n\}$. This category is generated by the coface and codegeneracy maps $d^i : [n-1] \to [n]$ and $s^j : [n+1] \to [n]$. This induces maps $d_i : X_n \to X_{n-1}$ and $s_j : X_n \to X_{n+1}$ for any simplicial set X whose n-simplices I denote by X_n.

1 Layer 1 - Simplicial Homotopy Theory



- A (closed) model structure on a category consists of three distinguised classes of maps, $\mathcal{C}$,$\mathcal{W}$,$\mathcal{F}$ satisfying a bunch of axioms which are abstractions of the traditional homotopy theory of spaces. There are model structures on each of the categories in the diagram above. Every model structure on a category gives rise to the "homotopy category" corresponding to the category. Objects of the homotopy category can be thought of as homotopy types. So in $\mathcal{H}\mathrm{o}(\mathcal{T}\mathrm{op})$, an object can be thought of as a space, upto homotopy equivalence.

- $\mathrm{s}\mathcal{S}\mathrm{et}$ which is the category of contravariant functors $\mathbf{\Delta}^{op} \to \mathcal{S}\mathrm{et}$ provides a *combinatorial model* for classical homotopy theory. The functors geometric realization $|\cdot|$ and singular functor $\mathbf{S}$ are the "original pair" of adjoint functors. The functor $\mathbf{S}$ takes a topological space T to the simplicial set $[n] \mapsto \hom(|\Delta^n|, T)$ These two functors set up a *Quillen equivalence* between $\mathrm{s}\mathcal{S}\mathrm{et}$ and $\mathcal{T}\mathrm{op}$ i.e., an equivalence of "homotopy theories".

- Instead of the category $\mathbf{\Delta}$, one can also consider other categories, such as the cube category and so on. Most generally, presheaves on any test category (in the sense of Grothendieck) gives rise to a model of standard homotopy theory.
- sAb is the category of contravariant functors $\mathbf{\Delta}^{op} \to \mathcal{A}b$. You can get an abelian group from a set easily by taking the free abelian group functor. If we do that in all simplical levels we get the functor $\mathbb{Z}$ in the diagram above. There is a forgetful functor u in the opposite direction. This pair of functors sets up a *Quillen adjunction* i.e., the functor u preserves cofibrations $\mathcal{C}$ and the functor $\mathbb{Z}$ preserves fibrations.
- We can make a (bounded) chain complex out of a simplicial abelian group A by taking the n-chains A_n to be the group of n-simplicies of A and the boundary map $\partial : A_n \to A_{n-1}$ to be $\partial = \sum_i (-1)^i d_i$ induced by the face maps induces by $d^i : [n-1] \to [n]$. This gives us a chain complex in $\mathcal{C}\mathrm{h}_+(\mathbb{Z})$. But we could do the same thing for simplcial R-modules for a unitary ring R.

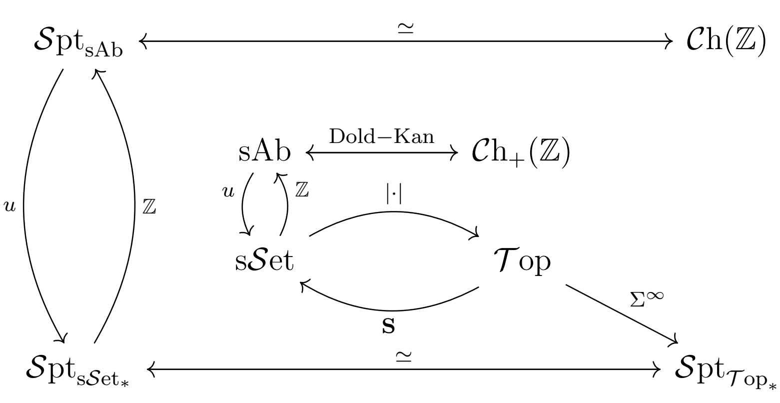
$$A_0 \xleftarrow{\partial} A_1 \xleftarrow{\partial} A_2 \ldots$$

The point is that both these categories sAb and $\mathcal{C}\mathrm{h}_+(\mathbb{Z})$ are equivalent under the *Dold-Kan correspondence* (note however, that the functor we gave above is not the one involved in the D-K correspondence)

Thus, the classical construction of (integral singular) homology of spaces is simply a "diagram chase", culminating in $H_*(\mathbb{Z}\mathbf{S}X) \simeq H_*(X, \mathbb{Z})$. Another note: the unit of the free-forgetful adjunction $h : X \to u\mathbb{Z}X$ is called the Hurewicz map.

2 Layer 2 - Stable Homotopy Theory

We now make the layer a bit more complicated by adding *spectra*.



- Recall that if $X, Y \in \mathcal{T}\mathrm{op}_*$ then the smash product of X and Y is

$$X \wedge Y = X \times Y / X \vee Y$$

It turns out that smashing a space $X \in \mathcal{T}\mathrm{op}$ with the circle S^1 is something you want to do:

$$\Sigma X \cong X \wedge S^1 \qquad \Sigma^k X \cong X \wedge S^k$$

Now for $X \in \mathcal{T}\mathrm{op}$, infinitely suspending it gives us a spectrum $E = \Sigma^\infty X \in \mathcal{S}\mathrm{pt}_{\mathcal{T}\mathrm{op}_*}$. A spectrum E (which is a list of spaces with suspension relation from one to the other) has *stable homotopy groups* $\pi_n(E)$. Computing the stable homotopy groups of the sphere spectrum:

$$S^0, S^1, S^1 \wedge S^1, \ldots$$

is the dominant question in modern algebraic topology.

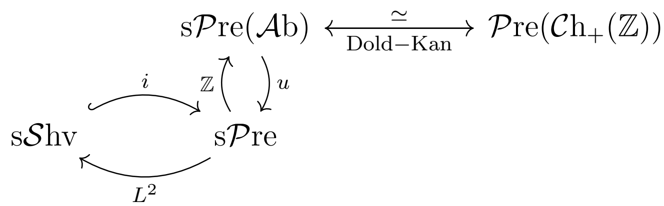
- Spectra in general have analogues in s$\mathcal{S}$et as *spectrum objects in pointed simplicial sets*, denoted $\mathcal{S}\mathrm{pt}_{\mathrm{s}\mathcal{S}\mathrm{et}_*}$ and the homotopy category (once model structures on them are defined) are equivalent.
- Given a spectrum object in simplicial sets, we could apply the free abelian group functor $\mathbb{Z}$ everywhere to map $S^1 \wedge E^n \mapsto \mathbb{Z}S^1 \otimes \mathbb{Z}E^n$ levelwise. This again behaves like the relationship between s$\mathcal{S}$et and sAb.
- Spectrum objects in (bounded) chain complexes are equivalent to unbounded chain complexes $\mathcal{C}$h. The Dold-Kan correspondence generates an equivalence of homotopy categories between $\mathcal{S}\mathrm{pt}_{\mathrm{sAb}}$ and $\mathcal{C}\mathrm{h}(R)$.

3 Layer 3 - Local Homotopy Theory

In topological spaces, one can exploit the poset structure of open subsets on topological spaces to define objects (such as sets of functions or more generally, algebraic objects) *locally*. This is made precise in the definition of *presheaves* and *sheaves*.

- A presheaf $\mathcal{F}$ of abelian groups on a topological space X assigns an abelian group $\mathcal{F}(U)$ to each open set U in X. If $V \subset U$ then there is a restriction map $\mathrm{res}_{VU} : \mathcal{F}(U) \to \mathcal{F}(V)$, satisfying certain compatibility conditions.(Precisely, a contravariant functor $(\mathrm{op}_S)^{\mathrm{op}} \to \mathcal{A}\mathrm{b}$.) One could do the same thing with the category $\mathcal{A}$b replaced with any other category.
- If a presheaf satisfies two further conditions named "locality" and "glueing", it is called a *sheaf*.
- For a fixed x, one says that elements $f \in \mathcal{F}(U)$ and $g \in \mathcal{F}(V)$ are equivalent at x if there is a neighbourhood $W \subset U \cap V$ of x with $\mathrm{res}_{WU}(f) = \mathrm{res}_{WV}(g)$ (both elements of $\mathcal{F}(W)$. The equivalence classes form the *stalk* $\mathcal{F}_x$ at x of the presheaf $\mathcal{F}$.

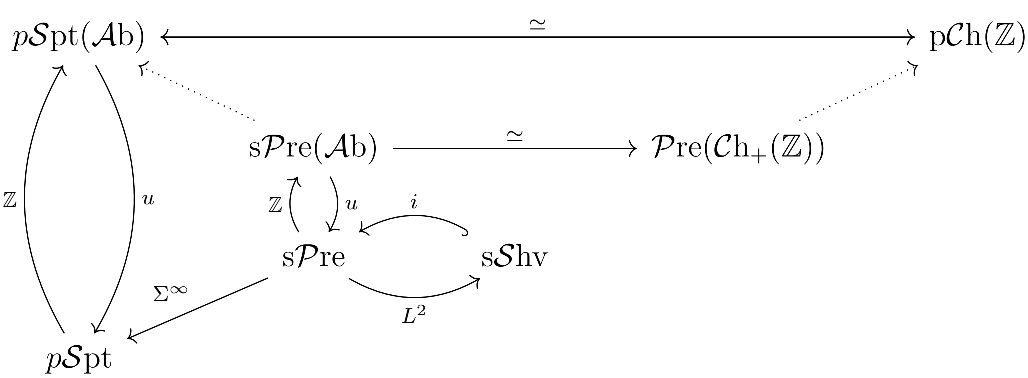
For whatever reason, we would like to consider *simplicial presheaves* s$\mathcal{P}$re and *simplicial sheaves* s$\mathcal{S}$hv, where $\mathcal{A}$b above is replaced by s$\mathcal{S}$et. To define the "local weak equivalences" on them, we exploit the underlying model structure on s$\mathcal{S}$et to say that $X \xrightarrow{f} Y$ is a local weak equivalence if it induces a weak equivalence of simplicial sets in every stalk $X_x \xrightarrow{f_x} Y_x$.



- From the sketch above, one can deduce model structures on s$\mathcal{S}$hv and s$\mathcal{P}$re the adjoint pair of functors $L^2 \dashv i$ forms a Quillen equivalence between these categories. So, in practice, it doesn't matter if we consider simplicial sheaves or presheaves. (Drives Algebraic Geometers nuts!)
- The impetus for doing this was to provide answers to old questions of Grothendieck and also, to solve "local-to-global" problems in algebraic K-theory (such as the Lichtenbaum-Quillen conjecture) in the early 80's that demanded a language like this.
- One can also define "simplicial abelian presheaves" and presheaves over bounded chain complexes. All of these things have local homotopy theories.
- The free-forgetful adjunction Quillen adjunction and the Dold-Kan correspondence is an equivalence of categories.

4 Layer 4 - Local Stable Homotopy Theory

Adding one more layer in complexity, we can look at stable versions of the categories we presented in the previous section.



- Just like with s$\mathcal{S}$et there was an associated theory of spectra, there is an associated theory of spectra for s$\mathcal{P}$re as well. The category of presheaves of spectra is denoted $p\mathcal{S}$pt. An object of this category is a functor $E : \mathcal{C}^{op} \to \mathcal{S}\text{pt}_{\text{s}\mathcal{S}\text{et}_*}$ which assigns an open set $U \to E(U)$ which is a spectrum object in simplicial sets.

- Spectra have stable homotopy groups so we can define local stable equivalence just by restricting to stalks. So $E \to F$ is a local stable equivalence if $E_x \to F_x$ is a stable equivalence of spectra for every x. From here on one can define a homotopy theory.

- We can also talk about spectrum objects in s$\mathcal{P}$re($\mathcal{A}$b), which I denote $p\mathcal{S}$pt($\mathcal{A}$b). $p\mathcal{S}$pt and $p\mathcal{S}$pt($\mathcal{A}$b) have a free and forgetful adjunction between them. Spectrum objects in presheaves of bounded chain complexes are simply presheaves of unbounded chain complexes $\mathcal{C}$h($\mathbb{Z}$). There is also an equivalence between $p\mathcal{S}$pt($\mathcal{A}$b) and p$\mathcal{C}$h($\mathbb{Z}$).

- The category $p\mathcal{S}$pt was the big context for problems in K-theory. It turns out that one can use $p\mathcal{S}$pt to study homotopy groups of spheres! This is because $p\mathcal{S}$pt is involved in elliptic cohomology theory and topological modular forms. This is where the modern theory is.

- Voevodsky discovered during his work on the Milnor conjecture that maybe smashing with S^1 to define the spectrum wasn't the right thing. So he defines a theory of presheaves of T-spectra so that the list of spaces E now are connected by bonding maps $T \wedge E^n \to E^{n+1}$ where $T = \mathbb{P}^1$, the projective line defined over a scheme or a field. Once we formally make $\mathbb{A}^n \to *$ a weak equivalence for each n within the context of the Nisnevich topology and talk about T-spectra we have the motivic stable category.

  Motivic cohomology can then be described in this context, and this was what proved the Bloch-Kato conjectures.

The pictures only get more complicated. This was the current state of knowledge by the end of the 90's, but people now don't look at model structures anymore, but instead look at ∞-categories.