DIFFERENTIAL GEOMETRY FINAL PROJECT

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1. Introduction

For this project, we outline the basic structure theory of Lie groups relating them to the concept of Lie algebras. Roughly, a Lie algebra encodes the "infinitesimal" structure of a Lie group, but is simpler, being a vector space rather than a nonlinear manifold.

At the local level at least, the Fundamental Theorems of Lie allow one to reconstruct the group from the algebra.

2. The Category of Local (Lie) Groups

The correspondence between Lie groups and Lie algebras will be *local* in nature, the only portion of the Lie group that will be of importance is that portion of the group close to the group identity 1. To formalize this locality, we introduce *local groups*:

Definition 2.1 (Local group). A *local topological group* is a topological space G, with an identity element $1 \in G$, a partially defined but continuous multiplication operation $\cdot : \Omega \to G$ for some domain $\Omega \subset G \times G$, a partially defined but continuous inversion operation $()^{-1} : \Lambda \to G$ with $\Lambda \subset G$, obeying the following axioms:

- Ω is an open neighbourhood of $G \times \{1\} \bigcup \{1\} \times G$ and Λ is an open neighbourhood of 1.
- (Local associativity) If it happens that for elements $g, h, k \in G$ $g \cdot (h \cdot k)$ and $(g \cdot h) \cdot k$ are both well-defined, then they are equal.
- (Identity) For all $g \in G$, $g \cdot 1 = 1 \cdot g = g$.
- (Local inverse) If $g \in G$ and g^{-1} is well-defined in G, then $g \cdot g^{-1} = g^{-1} \cdot g = 1$.

A local group is said to be *symmetric* if $\Lambda = G$, that is, every element $g \in G$ has an inverse in G. A *local Lie group* is a local group in which the underlying topological space is a smooth manifold and where the associated maps are smooth maps on their domain of definition.

Clearly, every topological group is a local group and every Lie group is a local Lie group. We will call a Lie group *global* Lie group to distinguish it from a local Lie group. One can also consider local discrete groups in which the topology is just the discrete topology. A class of examples of local Lie groups comes from restricting a global Lie group to a neighbourhood of the identity.

Definition 2.2 (Restriction). Given a local group G and an open neighbourhood U of $1 \in G$. We define the *restriction* $G|_U$ of G to U to have domains $\Omega|_U = \{(g,h) \in \Omega : g, h, g \cdot h \in U\}$ and $\Lambda|_U = \{g \in \Lambda : g, g^{-1} \in U\}$ and having multiplication and inverse maps to be the multiplication and inverse maps of G restricted to U.

Thus, for instance, one can take the Euclidean space $\mathbb{R}^d$, and restrict it to a ball B centred at the origin, to obtain an additive local group $\mathbb{R}^d|_B$. In this group, two elements x, y in B have a well-defined sum $x + y$ only when their sum in $\mathbb{R}^d$ stays inside B. Intuitively, this local group behaves like the global group $\mathbb{R}^d$ as long as one is close enough to the identity element 0, but as one gets closer to the boundary of B, the group structure begins to break down.

Example 2.3. Let G be a global or local Lie group of some dimension d, and let $\phi : U \to V$ be a smooth coordinate chart from a neighbourhood U of the identity 1 in G to a neighbourhood V of the origin 0 in $\mathbb{R}^d$, such that ϕ maps 1 to 0. Then we can define a local group $\phi_* G|_U$ which is the set V (viewed as a smooth submanifold of $\mathbb{R}^d$) with the local group identity 0, the local group multiplication law $*$ defined by the formula

$$x * y := \phi(\phi^{-1}(x) \cdot \phi^{-1}(y))$$

defined whenever $\phi^{-1}(x), \phi^{-1}(y), \phi^{-1}(x) \cdot \phi^{-1}(y)$ are well-defined and lie in U, and the local group inversion law $()^{*-1}$ defined by the formula

$$x^{*-1} := \phi(\phi^{-1}(x)^{-1})$$

defined whenever $\phi^{-1}(x), \phi^{-1}(x)^{-1}$ are well-defined and lie in U. One easily verifies that $\phi_* G|_U$ is a local Lie group. We will sometimes denote this local Lie group as $(V, *)$, to distinguish it from the additive local Lie group $(V, +)$ arising by restriction of $\mathbb{R}^d$ to V.

Definition 2.4 (Homorphisms). A *continuous homomorphism/smooth homomorphism* of local Lie groups G and H is a continuous/smooth function ϕ from G to H such that

- $\phi(1_G) = 1_H$
- If $g \in G$ is such that g^{-1} is well defined in G, then $\phi(g)^{-1}$ is well defined in H and equals $\phi(g^{-1})$.
- If $g, h \in G$ are such that $g \cdot h$ is well defined in G, then $\phi(g) \cdot \phi(h)$ is well defined in H and equals $\phi(g \cdot h)$

It is easy to see that the composite of two homomorphisms is also a homomorphism. This defines the category of Local Groups and the category of Local Lie Groups.

3. Basic Differential Geometry

To define the Lie algebra of a Lie group, we must first quickly recall some basic notions from differential geometry associated to smooth manifolds.

Definition 3.1 (Tangent Space). If M is a d-dimensional manifold, at each $x \in M$ we can define the tangent space to M at x, denoted $T_x(M)$, as the set of all equivalence classes of maps $\gamma : [0,1] \to M$ with $\gamma(0) = x$ where two maps $gamma_1$ and $gamma_2$ are equivalent if and only if

$$\frac{d}{dt}\phi(\gamma_1(t))|_{t=0} = \frac{d}{dt}\phi(\gamma_2(t))|_{t=0}$$

where $\phi : U \to V$ is a coordinate chart of G defined on a neighbourhood of x. It is easy to see (by the chain rule) that this equivalence relation does not depend on the particular choice of the chart ϕ. One can identify $T_x(M)$ with $\mathbb{R}^d$ by associating each γ with $\frac{d}{dt}\phi(\gamma_1(t))|_{t=0}$. This gives $T_x(M)$ the structure of a vector space of dimension d. Elements of $T_x(M)$ are called *tangent vectors* of M at x. We will denote the element $[\gamma]$ of $T_x(M)$ by $\gamma'(0)$.

Definition 3.2 (Tangent Bundle). The space $TM := \bigcup_{x\in M}\{x\} \times T_x(M)$ of pairs (x, v) where x is a point in M and v is a tangent vector to M at x is called the *tangent bundle*.

Definition 3.3 (Derivative). If $\Phi : M \to N$ is a smooth map of manifolds, we can define the derivative $D\Phi : TM \to TN$, by setting

$$D(\Phi(x, \gamma'(0))) := (\Phi(x), (\Phi \circ \gamma)'(0))$$

for all continuously differentiable curves $\gamma : [0,1] \to M$ with $\gamma(0) = x$ for some $x \in M$. We also write $(\Phi(x), D\Phi(x)(v))$ for $D\Phi(x, v)$, so that for each $x \in M$, $D\Phi(x)$ is a map from T_xM to $T_{\Phi(x)}N$. One can easily verify that this latter map is linear. We observe the chain rule

$$D(\Psi \circ \Phi) = (D\Psi) \circ (D\Phi) \tag{1}$$

for any smooth maps $\Phi : M \to N$, $\Psi : N \to O$.

Once we have the notion of a tangent bundle, we can define the notion of a smooth vector field.

Definition 3.4 (Vector fields). A *smooth vector field* on M is a smooth map $X : M \to TM$ which is a right inverse for the projection map $\pi : TM \to M$. Thus, X maps an $x \in M$ to (x, v) where $v \in T_x(M)$ is a tangent vector to M at x. By abuse of notation, we shall denote denote v as $X(x)$. Denote by $\Gamma(TM)$ the $C^\infty(M)$-module of all vector fields over M. Where $C^\infty(M)$ is the space of all smooth maps from $M \to \mathbb{R}$.

Definition 3.5 (Directional Derivative). Given a smooth function $f \in C^\infty(M)$ and a smooth vector field $X \in \Gamma(TM)$, define the *directional derivative* $\nabla_x(f) \in C^\infty(M)$ of f along X by the formula

$$\nabla_X(f)(x) := \frac{d}{dt}f(\gamma(t))|_{t=0}$$

whenever $\gamma : [0,1] \to M$ is a smooth function with $\gamma(0) = x$ and $\gamma'(x) = X(x)$. One easily verifies that $\nabla_X(f)$ is well defined and is an element of $C^\infty(M)$.

There is a correspondence between smooth vector fields and *derivations* of $C^\infty(M)$: If M is a smooth manifold, a *derivation* on $C^\infty(M)$ is a smooth linear map which satisfies the Liebniz rule on $C^\infty(M)$. The correspondence says that given a vector field X on a smooth manifold M, we get a derivation ∇_X out of it, and conversely, given any derivation d of $C^\infty(M)$, it is possible to get a unique smooth vector field X such that $d = \nabla_X$.

Proposition 3.6. *Let $d_1, d_2 : A \to A$ be two derivations on an algebra A. Then, the commutator*

$$[d_1, d_2] := d_1 \circ d_2 - d_2 \circ d_1$$

is also a derivation.

Definition 3.7 (Lie bracket). We can define the Lie bracket $[X, Y]$ of two vector fields $X, Y \in \Gamma(TM)$ as

$$\nabla_{[X,Y]} := [\nabla_X, \nabla_Y]$$

So the Lie bracket of two vector fields X, Y is that unique vector field which corresponds to the commutator function $[\nabla_X, \nabla_Y] \in C^\infty(M)$

Definition 3.8 (Lie algebra). A real Lie algebra is a real vector space V with a bilenear map $[,] : V \times V \to V$ which is antisymmetric (that is, $[X, Y] = -[Y, X]$) and obeys the Jacobi identity:

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

for all $X, Y, Z \in V$.

The space of all vector fields on a smooth manifold M, $\Gamma(TM)$ is a Lie algebra with the Lie bracket is a Lie algebra.

4. The Lie algebra of a Lie group

Let G be a (global) Lie group. So by definition G is a smooth manifold so we can define the tangent bundle TG and smooth vector fields $X \in \Gamma(TG)$ as in the previous section. In particular, we can define the tangent space of G at the identity $T_1(G)$.

Definition 4.1 (Left invariant vector fields). Fix a Lie group G. The map $\rho_g^{left} : x \mapsto gx$ is a smooth map, and so we can look at it's derivative $D\rho_g^{left} : TG \to TG$. Now a vector field $X : G \to TG$ is called *left-invariant* if and only if the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\rho_g^{left}} & G \\ \big\downarrow X & & \big\downarrow X \\ TG & \xrightarrow{D\rho_g^{left}} & TG \end{array}$$

Equivalently, we have $D\rho_g^{left} \circ X = X \circ \rho_g^{left}$

We have the following two facts:

- If X is a left invariant vector field, the commutativity of the above diagram gives us that for any $a \in G$, $X \circ \rho_a^{left}(1) = X(a) = (D\rho_a^{left}) \circ X(1)$. So if we know $X(1) \in T_1(G)$, we can recover the entire vector field X.
- The commutator of two left invariant vector fields is again left invariant.

From the above two facts, we can identify the set of left invariant vector fields $\mathfrak{X}_L(G) \subset \Gamma(TG)$ on a Lie group with the tangent space at the identity $T_1(G)$. The Lie bracket structure on $\Gamma(TG)$ then induces a Lie bracket on $T_1(G)$, which is also denoted $[,]$.

Definition 4.2 (Lie algebra). For a Lie group G, we have an isomorphism between the tangent space of G at the identity and the set of all left-invariant vector fields $\mathfrak{X}_L(G) \subset \Gamma(TG)$. The Lie bracket on $\Gamma(TG)$ and the isomorphism induces a Lie bracket on $T_1((G)$. The vector space $T_1(G)$ along with the induced Lie bracket $[,]$ is called the Lie algebra of the Lie group G. It is denoted $\mathfrak{g}$.

Informally, an element x of the Lie algebra $\mathfrak{g}$ is associated with an infinitesimal perturbation $1 + \epsilon x + O(\epsilon^2)$ of the identity in the Lie group G. This intuition can be formalized fairly easily in the case of matrix Lie groups such as $GL_n(\mathbb{C})$. The isomorphism $L : \mathfrak{g} \to \mathfrak{X}_L(G)$ carries $X \mapsto L_X$. Where $L_X(a) = D(\rho_a^{left}) \circ X$.

We have seen that every global Lie group gives rise to a Lie algebra. One can also associate Lie algebras to *local* Lie groups. In the converse direction, it is also true that every finite-dimensional Lie algebra can be associated to either a local or a global Lie group; this is known as *Lie's third theorem*. However, this theorem is somewhat tricky to prove (particularly if one wants to associate the Lie algebra with a global Lie group), requiring the non-trivial algebraic tool of Ado's theorem. Without resorting to Ado's theorem, we can prove that **every finite dimensional Lie algebra is isomorphic to the Lie algebra of a *local* Lie group**. For this, we need to develop further machinery.

5. The exponential map

Given a smooth manifold M and a vector field $X \in \Gamma(M)$, we define an *integral curve* of X to be a smooth curve $c : J \to M$ for an interval $J \subseteq \mathbb{R}$ such that $\frac{d}{dt}c(t) = X(c(t))$ for all $t \in J$. It can be shown that for a vector field $X \in \Gamma(M)$ and any $x \in M$ there is an open interval J_x containing 0 and an integral curve $c_x : J_x \to M$ for X with $c_x(0) = x$. If J_x is maximal, c_x is unique.

Definition 5.1 (Flow of a vector field). Let $X \in \Gamma(M)$ be a vector field. We write $\mathrm{Fl}_t^X(x) = \mathrm{Fl}(t, x) := c_x(t)$ where $c_x : J_x \to M$ is the maximally defined integral curve with $c_x(0) = x$ as mentioned above.

Definition 5.2 (One-parameter subgroups). Let G be a Lie group with Lie algebra $\mathfrak{g}$. A 1-parameter subgroup of G is a Lie group homomorphism $\alpha : (\mathbb{R}, +) \to G$, that is, a smooth curve α in G such that $\alpha(s + t) = \alpha(s)\alpha(t)$ and hence $\alpha(0) = 1$.

It is a theorem that 1-parameter subgroups are in correspondence with flows associated with left-invariant vector fields.

Definition 5.3 (Exponential map). The exponential mapping $\exp : \mathfrak{g} \to G$ of a Lie group is defined by

$$\exp X = \mathrm{Fl}^{L_X}(1, 1_G) = \alpha_X(1)$$

where α_X is the 1-parameter subgroup of G with $\frac{d}{dt}\alpha_X(t)|_0 = X$ and L_X is a left-invariant vector field $L_X(a) = D(\rho_a^{left}) \circ X$.

So, to summarize, given an $X \in \mathfrak{g}$ the isomorphism L gives rise to a left-invariant vector field $L_X \in \mathfrak{X}_L(G)$. We then take the 1-parameter subgroup of the associated flow of L_X and set the exponential of X to be that 1-parameter subgroup evaluated at 1.

Now given a Lie group G, we can define the smooth map of conjugation by a group element $a \in G$: $\text{conj}_a : x \mapsto axa^{-1}$.

Definition 5.4 (Adjoint Representation). Given a Lie group G and an element $a \in G$. We define the linear mapping $\text{Ad}(a) = D\text{conj}_a(1) : T_1(G) = \mathfrak{g} \to T_1(G) = \mathfrak{g}$ to be the derivative of the conjugation map at the group identity. So $\text{Ad}(a) \in GL(\mathfrak{g})$. Furthermore, one can show that $\text{Ad} : G \to GL(\mathfrak{g})$ is a representation, called the *adjoint representation* of G, that is, it is actually a group homomorphism.

Now the *adjoint representation* of the Lie algebra $\mathfrak{g}$ is the derivative of the above representaiton at the identity:

$$\text{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \qquad \text{ad} := D(\text{Ad})(1)$$

Now it can be shown that for all $X, Y \in \mathfrak{g}$, we have $\text{ad}(X)Y = [X, Y]$

So, summarizing, we have the following:

$$\begin{aligned} \text{Ad} : G &\to GL(\mathfrak{g}) \\ a &\mapsto (\text{Ad}(a) = D(\text{conj}_a)(1)) : \mathfrak{g} \to \mathfrak{g} \end{aligned}$$

And

$$\begin{aligned} \text{ad} = D(\text{Ad})(1) : \mathfrak{g} &\to \mathfrak{gl}(\mathfrak{g}) \\ X &\mapsto (\text{ad}(X)Y = [X, Y]) \end{aligned}$$

We have the following proposition.

Proposition 5.5. *Given a Lie group G with Lie algebra $\mathfrak{g}$. Let $Ad : G \to GL(\mathfrak{g})$ be the adjoint representation of G and let $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ be the adjoint representation of the $\mathfrak{g}$. Also let $\exp^G : \mathfrak{g} \to G$ be the exponential map. Then we have:*

$$Ad \circ \exp^G = \exp^{GL(\mathfrak{g})} \circ ad$$

So for $X, Y \in G$

$$Ad_X(Y) := Ad(\exp^G X).Y = \sum_{k=0}^{\infty} \frac{1}{k!}(ad_X)^k.Y = \exp(ad_X)Y$$

We have everything to state the Baker-Campbell-Hausdorff formula.

Let $\mathfrak{g}$ be a finite dimensional real Lie algebra. Given two sufficiently small elements x, y of $\mathfrak{g}$, define

$$R_y(x) = x + \int_0^1 F_R(\text{Ad}_x \text{Ad}_{ty}) y dt$$

where $\text{Ad}_x := \exp(\text{ad}_x)$, $\text{ad}_x : \mathfrak{g} \to \mathfrak{g}$ is the adjoint map $\text{ad}_x(y) = [x, y]$ and F_R is the function $\frac{z \log z}{z-1}$, which is analytic for z near 1. Similarly we can define

$$L_x(y) := y + \int_0^1 F_L(\mathrm{Ad}_{tx}\mathrm{Ad}_y)x\ dt$$

Where $F_L(z) = \frac{\log z}{z-1}$

Theorem 5.6 (Baker-Campbell-Hausdorff). *Let G be a finite dimensional real Lie group with Lie algebra $\mathfrak{g}$. For x, y near 1 in $\mathfrak{g}$, we have $\exp x \exp y = \exp C(x,y)$ where $C(x,y) = R_y(x) = L_x(y)$. We also have a formal power series expansion for $C(x,y)$.*

$$C(x,y) = x + y + \frac{1}{2}[x,y] + \frac{1}{12}([x,[x,y]] + [y,[y,x]]) + \dots$$

6. Proving the Local form of Lie's third theorem

It is possible to use the Baker-Campbell-Hausdorff formula and conclude (the local form of) Lie's third theorem, that every finite-dimensional Lie algebra over the reals is isomorphic to the Lie algebra of some local Lie group. In particular, given a Lie algebra $\mathfrak{g}$ as above, we can endow a neighbourhood of 1 in $\mathfrak{g}$ with a a structure of a local Lie group by giving it a multiplication

$$x * y := R_y(x) = L_x(y) \tag{2}$$

for sufficiently small x, y and the inverse operation by $x^{-1} := -x$. It is a theorem of Dynkin that this operation has an explicit formal sum:

$$x * y = x + \sum_{n=0}^{\infty} \frac{(-1)^m}{n+1} \sum_{\substack{r_i, s_i \geq 0 \\ (r_i, s_i) \neq (0,0)}} \frac{(\mathrm{ad}_y)^{r_1}(\mathrm{ad}_x)^{s_1} \dots (\mathrm{ad}_y)^{r_n}(\mathrm{ad}_x)^{s_n}}{r_1! s_1! \dots r_n! s_n! (r_1 + \dots + r_n + 1)} y$$

The above formula is called the *BCH formula in Dynkin form.*

Since we start with an arbitary real Lie algebra, we need additional hypothesis on the convergence of the group law we have just defined. For this, we use the following theorem.

Theorem 6.1 (Dynkin). *Let $\mathfrak{g}$ be a finite-dimensional real lie algebra with a norm $\|.\|$ with a real number $\mu > 0$ such that*

$$\|[x,y]\| \leq \mu \|x\| \|y\|$$

for all $x, y \in \mathfrak{g}$. Then the domain of absolute convergence of the BCH series in Dynkin form contains the open set

$$C = \left\{ (x,y) \in \mathfrak{g} \times \mathfrak{g} : \|x\| + \|y\| < \frac{1}{\mu} \log 2 \right\}$$

We have the following result:

Proposition 6.2. *Assume that $*$ defined above makes $(\mathfrak{g}, *)$ into a local Lie group. We can associate a Lie algebra $\mathfrak{g}'$ to this local Lie group $(\mathfrak{g}, *)$. We have that this Lie algebra is the same as the Lie algebra $\mathfrak{g}$.*

Proof. Note first that $T_1(\mathfrak{g}') = \mathfrak{g}$. So we are done if we show that the the induced Lie bracket $[,]^*$ and the inherited Lie bracket $[,]$ (which is just a bilinear form) are the same. Recall that for a local Lie group $(\mathfrak{g}, *)$ we define $[,]^*$ as follows:

$$[x, y]^* = \frac{\partial^2}{\partial t \partial s}[g(t), h(s)]^C|_{t=s=0}$$

where g, h are differentiable paths from $g : I_g \to \mathfrak{g}$ and $h : I_h \to \mathfrak{g}$ on $\mathfrak{g}$ such that $0 \in I_g$, $0 \in I_h$, $g(0) = 0 = h(0)$ and $g'(0) = x$ and $h'(0) = y$ and $[g(t), h(s)]^C = g(t)*h(s)*g(t)^{-1}*h(s)^{-1}$ is the group commutator. Set

$$A = g(t) * h(s) = g(t) + h(s) + \frac{1}{2}[g(t), h(s)] + \dots$$
$$B = g(t)^{-1} * h(s)^{-1} = g(t)^{-1} + h(s)^{-1} + \frac{1}{2}[g(t)^{-1}, h(s)^{-1}] + \dots$$

where the dots indicate higher order terms. Assuming associativity of $*$, we now want to compute $\frac{\partial^2}{\partial t \partial s} A * B$.

Fact 1: $\frac{\partial^2}{\partial t \partial s}[g(t), h(s)] = [g'(t), h'(s)]$. This follows from the linearity of the derivative.

Fact 2: For any bilinear form on a real vector space $[,]$, one has $[0, x] = [y, 0] = 0$ for all $x, y \in V$. Immediate from the definition of bilinearity.

Fact 3: $0 * x = y * 0 = 0$ for all $x, y \in \mathfrak{g}$.

Claim: Because of facts 1 and 2, the higher order terms in the evaluation of the partial derivative all vanish. As an example, the next term in the above series is $\frac{1}{12}([X, [X, Y]] + [Y, [Y, X]])$. Now for example, if we set $X = g(t)$ and $Y = h(s)$, the third term vanishes because after calculating the partials and evaluating them at $s = t = 0$.

So the only term of interest is $D = \frac{1}{2}[g(t), h(s)]$ and $E = \frac{1}{2}[g(t)^{-1}, h(s)^{-1}]$. We then have:

$$\frac{1}{2}\frac{\partial^2}{\partial t \partial s}[g(t), h(s)] = \frac{1}{2}[g'(t), h'(s)]|_{s=t=0} = \frac{1}{2}[x, y]$$
$$\frac{1}{2}\frac{\partial^2}{\partial t \partial s}[g(t)^{-1}, h(s)^{-1}] = \frac{1}{2}\left[-g(t)^{-2} * g'(t), -h(s)^{-2} * h'(s)\right]|_{s=t=0}$$
$$= \frac{1}{2}[0 + x, 0 + y] = \frac{1}{2}[x, y]$$

Where we have used Fact 3 in establishing the second chain of equalities. Now since $A*B = A + B + \dots$, we finally get that

$$[x, y]^* = \frac{1}{2}[x, y] + \frac{1}{2}[x, y] = [x, y]$$

□

So all we need to show is that the group operation $*$ is well defined and associative. The key is the following proposition.

Proposition 6.3. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then for sufficiently small x, y, z, one has*

$$L_y(R_z(x)) = R_z(L_y(x)) \tag{3}$$

Assuming this proposition, we can set $x = 0$ in (3) and use the identities $R_z(0) = z$ and $L_y(0) = y$ (seen to be true from the formal power series expansions), we can conclude that for small enough y, z we have $L_y(z) = R_z(y)$, thus showing well-definedness. Then inserting (2) into (3) we obtain the desired local associativity since $L_y(R_z(x)) = y*(x*z) = R_z(L_y(x)) = (y*x)*z$

So we just need to prove the proposition.

- First, note that $\mathrm{ad} : x \mapsto \mathrm{ad}_x$ is a Lie algebra homomorphism from $\mathfrak{g}$ to the Lie algebra $\mathfrak{gl}(\mathfrak{g})$. This is because if it is a Lie algebra homomorphism, we have $[\mathrm{ad}_x, \mathrm{ad}_y](z) = \mathrm{ad}_{[x,y]}(z)$, which is $[x,[y,z]] - [y,[x,z]] = [[x,y],z]$, which is the Jacobi identity.
- Now since $\mathfrak{gl}(\mathfrak{g})$ is the Lie algebra of $GL(\mathfrak{g})$,a Lie group, we have the Baker-Campbell-Hausdorff formula for $\mathfrak{gl}(\mathfrak{g})$, which says, in particular, that
$$\log(\exp(\mathrm{ad}_x)\exp(\mathrm{ad}_y)) = R_{\mathrm{ad}_y}(\mathrm{ad}_x) = L_{\mathrm{ad}_x}(\mathrm{ad}_y)$$
- Since ad is a Lie algebra homomorphism,
$$R_{\mathrm{ad}_y}(\mathrm{ad}_x) = \mathrm{ad}_{R_y(x)}$$
and also
$$L_{\mathrm{ad}_x}(\mathrm{ad}_y) = \mathrm{ad}_{L_x(y)}$$
Exponentiating on both sides since $\mathrm{Ad}_x = \exp(\mathrm{ad}_x)$, we get
$$\mathrm{Ad}_x\mathrm{Ad}_y = \mathrm{Ad}_{R_y(x)} = \mathrm{Ad}_{L_x(y)}$$
- Using the above, we can rewrite
$$R_y(x) = x + \int_0^1 F_R(\mathrm{Ad}_{R_{ty}(x)})y\,dt$$
$$L_x(y) = y + \int_0^1 F_L(\mathrm{Ad}_{L_{tx}(y)})x\,dt$$
- This leads to the *radial homogenity identities*: for $x, y \in \mathfrak{g}$ sufficiently small and $0 \leq s, t \leq 1$.
$$R_{(s+t)y}(x) = R_{sy}(R_{ty}(x))$$
and
$$L_{(s+t)x}(y) = L_{sx}(L_{tx}(y))$$
The computation for this is in the notes.
- Because of the above identities, it will now suffice to prove the approximate commutativity law:
$$L_y(R_z(x)) = R_z(L_y(x)) + O(|y|^2|z|) + O(|y||z|^2) \tag{4}$$
for all small x, y, z. Indeed, this law implies that
$$L_{y/n} \circ R_{z/n} = R_{z/n} \circ L_{y/n} + O(1/n^3) \tag{5}$$

- After sufficient number of iterations of (5), we end up with the following

$$L_y \circ R_z = R_z \circ L_y + O(1/n)$$

The sketch of this is in the notes. The idea is to evaluate the expression repeatedly at $R_{z/n}(x)$ and $L_{y/n}(x)$ to get the required. Notice that the error is larger in the iterated expression. Making $n \to \infty$, we obtain the required.
- It remains to prove (4). When $y = 0$, then L_y is the identity map and the claim is trivial; similarly if $z = 0$. Fix x and look at

$$F(a, b) = L_{ay}(R_{bz}(x)) - R_{bz}(L_{ay}(x))$$

By Taylor expansion of F about (0,0), it thus suffices to establish the infinitesimal commutativity law

$$\frac{\partial}{\partial a}\frac{\partial}{\partial b}L_{ay}(R_{bz}(x))|_{a=b=0} = \frac{\partial}{\partial a}\frac{\partial}{\partial b}R_{bz}(L_{ay}(x))|_{a=b=0}.$$

- From the fundamental theorem of calculus one has

$$L_{ay}(w) := w + \int_0^1 F_L(\mathrm{Ad}_{tay}\mathrm{Ad}_w)ay\ dt$$

$$\frac{\partial}{\partial a}L_{ay}(w)|_{a=0} = F_L(\mathrm{Ad}_w)y$$

for any fixed y, w, and similarly

$$\frac{\partial}{\partial b}R_{bz}(v)|_{b=0} = F_R(\mathrm{Ad}_v)z.$$

And so it suffices to show that

(6) $$\frac{\partial}{\partial b}F_L(\mathrm{Ad}_{R_{bz}(x)})y|_{b=0} = \frac{\partial}{\partial a}F_R(\mathrm{Ad}_{L_{az}(x)})z|_{a=0}$$

- It will be more convenient to work with the reciprocals F_L^{-1}, F_R^{-1} of the functions F_L, F_R. Recall the general matrix identity

$$\frac{d}{dt}A^{-1}(t) = -A^{-1}(t)A'(t)A^{-1}(t)$$

for any smoothly varying invertible matrix function $A(t)$ of a real parameter t. Using this, we write the left hand side of (6) as

$$-F_L(\mathrm{Ad}_x)(\frac{\partial}{\partial b}F_L^{-1}(\mathrm{Ad}_{R_{bz}(x)})|_{b=0})F_L(\mathrm{Ad}_x)y$$

- If we write $Y := F_L(\mathrm{Ad}_x)y$ and $Z := F_R(\mathrm{Ad}_x)z$, then from Taylor expansion about b and a as in two items before, we have

$$R_{bz}(x) = x + bZ + O(|b|^2)$$

$$L_{ay}(x) = x + aY + O(|a|^2)$$

Putting everything together, (6) now becomes:

(7) $$-F_L(\mathrm{Ad}_x)(\frac{\partial}{\partial b}F_L^{-1}(\mathrm{Ad}_{x+bZ})|_{b=0})Y = -F_R(\mathrm{Ad}_x)(\frac{\partial}{\partial a}F_R^{-1}(\mathrm{Ad}_{x+aY})|_{a=0})Z$$

Because $F_R(\mathrm{Ad}_x) = \mathrm{Ad}_x F_L(\mathrm{Ad}_x)$, it suffices to show:

$$(\frac{\partial}{\partial b} F_L^{-1}(\mathrm{Ad}_{x+bZ})|_{b=0})Y = \mathrm{Ad}_x(\frac{\partial}{\partial a} F_R^{-1}(\mathrm{Ad}_{x+aY})|_{a=0})Z. \tag{8}$$

- Now, we write (how?)

$$F_L^{-1}(\mathrm{Ad}_x) = \int_0^1 \mathrm{Ad}_{tx}\ dt$$

and

$$F_R^{-1}(\mathrm{Ad}_x) = \int_0^1 \mathrm{Ad}_{-tx}\ dt$$

and thus expand (8) as

$$\int_0^1 (\frac{\partial}{\partial b}\mathrm{Ad}_{tx+tbZ})|_{b=0} Y\ dt = \mathrm{Ad}_x \int_0^1 (\frac{\partial}{\partial b}\mathrm{Ad}_{-tx-taY})|_{a=0} Z\ dt. \tag{9}$$

We write Ad as the exponential of ad. Using the Duhamel matrix identity

$$\frac{d}{dt}\exp(A(t)) = \int_0^1 \exp(sA(t))A'(t)\exp((1-s)A(t))\ dt$$

for any smoothly varying matrix function $A(t)$ of a real variable t, together with the linearity of ad, we see that

$$(\frac{\partial}{\partial b}\mathrm{Ad}_{tx+tbZ})|_{b=0} = \int_0^1 \mathrm{Ad}_{stx} t \mathrm{ad}_Z \mathrm{Ad}_{(1-s)tx}\ ds$$

and similarly

$$(\frac{\partial}{\partial b}\mathrm{Ad}_{-tx-taY})|_{a=0} = -\int_0^1 \mathrm{Ad}_{-stx} t \mathrm{ad}_Y \mathrm{Ad}_{-(1-s)tx}\ ds.$$

Collecting terms, our task is now to show that

$$\int_0^1\int_0^1 \mathrm{Ad}_{stx}\mathrm{ad}_Z\mathrm{Ad}_{(1-s)tx} Y\ tdsdt = -\int_0^1\int_0^1 \mathrm{Ad}_{(1-st)x}\mathrm{ad}_Y\mathrm{Ad}_{-(1-s)tx} Z\ tdsdt. \tag{10}$$

- For any $x \in \mathfrak{g}$, the adjoint map $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$ is a derivation in the sense that:

$$\mathrm{ad}_x[y,z] = [\mathrm{ad}_x y, z] + [y, \mathrm{ad}_x z],$$

thanks to the Jacobi identity. Exponentiating, we conclude that

$$\mathrm{Ad}_x[y,z] = [\mathrm{Ad}_x y, \mathrm{Ad}_x z]$$

(thus each Ad_x is a Lie algebra homomorphism) and thus

$$\mathrm{Ad}_x \mathrm{ad}_y = \mathrm{ad}_{\mathrm{Ad}_x y}\mathrm{Ad}_x \implies \mathrm{Ad}_x\mathrm{ad}_y\mathrm{Ad}_x^{-1} = \mathrm{ad}_{\mathrm{Ad}_x y}$$

- Using this, we can simplfy (10) to look like

$$\int_0^1\int_0^1 \mathrm{ad}_{\mathrm{Ad}_{stx}Z}\mathrm{Ad}_{tx}Y\ tdsdt = -\int_0^1\int_0^1 \mathrm{ad}_{\mathrm{Ad}_{(1-st)x}Y}\mathrm{Ad}_{(1-t)x}Y\ tdsdt$$

which we can rewrite as

$$\int_0^1 \int_0^1 [\mathrm{Ad}_{stx} Z, \mathrm{Ad}_{tx} Y]\ tdsdt = -\int_0^1 \int_0^1 [\mathrm{Ad}_{(1-st)x} Y, \mathrm{Ad}_{(1-t)x} Y]\ tdsdt.$$

But by an appropriate change of variables (and the anti-symmetry of the Lie bracket), both sides of this equation can be written as

$$\int_{0 \leq a \leq b \leq 1} [\mathrm{Ad}_{ax} Z, \mathrm{Ad}_{bx} Y]\ dadb$$

and the claim follows!

The above argument shows that every finite-dimensional Lie algebra $\mathfrak{g}$ can be viewed as arising from a local Lie group G. It is natural to then ask if that local Lie group (or a sufficiently small piece thereof) can in turn be extended to a global Lie group $\tilde{G}$. The answer to this is affirmative, as was first shown by Cartan. It is unlikely that there is a proof of this result that does not either use Ados theorem, the proof method of Ados theorem (in particular, the structural decomposition of Lie algebras into semisimple and solvable factors), or some facts about group cohomology (particularly with regards to central extensions of Lie groups) which are closely related to the structural decompositions just mentioned. (As noted by Serre, though, a certain amount of this sort of difficulty in the proof may in fact be necessary, given that the global form of Lies third theorem is known to fail in the infinite-dimensional case.)